

# Isothermal Navier-Stokes Equations and Radon Transform

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## Abstract

In the paper we prove the existence results for initial-value boundary value problems for compressible isothermal Navier-Stokes equations. We restrict ourselves to 2D case of a problem with no-slip condition for nonstationary motion of viscous compressible isothermal fluid. However, the technique of modeling and analysis presented here is general and can be used for 3D problems.

Key words: Navier–Stokes equations, compressible fluids, Radon transform

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## 1 Introduction

### 1.1 Problem formulation

Suppose a viscous compressible fluid occupies a bounded domain  $\Omega \subset \mathbb{R}^2$ . The state of the fluid is characterized by the macroscopic quantities: the density  $\varrho(x, t)$  and the velocity  $\mathbf{u}(x, t)$ . The problem is to find  $\mathbf{u}(x, t)$  and  $\varrho(x, t)$  satisfying the following equations and boundary conditions in the cylinder

$$Q_T = \Omega \times (0, T).$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \varrho = \operatorname{div} \mathbb{S}(\mathbf{u}) + \varrho \mathbf{f} \quad \text{in } Q_T, \quad (1a)$$

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } Q_T, \quad (1b)$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (1c)$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \varrho(x, 0) = \varrho_0(x) \quad \text{in } \Omega. \quad (1d)$$

Here, the vector field  $\mathbf{f}$  denotes the density of external mass forces, the viscous stress tensor  $\mathbb{S}(\mathbf{u})$  has the form

$$\mathbb{S}(\mathbf{u}) = \nu_1 (\nabla \mathbf{u} + \nabla \mathbf{u}^\top) + \nu_2 \operatorname{div} \mathbf{u} \mathbb{I}, \quad (1e)$$

in which the viscosity coefficients satisfy the inequalities  $\nu_1 > 0$ ,  $\nu_1 + \nu_2 \geq 0$ . It is necessary to notice that problem (1) is the simplest multidimensional boundary value problem for the compressible Navier-Stokes equations. In 1986 Padula, see [8], formulated the result on existence of a weak solution to problem (1), but the proof presented was incomplete, see [9]. The first nonlocal results concerning the mathematical theory of compressible Navier-Stokes equations are due to P.-L. Lions. In monograph [6] he established the existence of a renormalized solution to nonstationary boundary value problem for the Navier-Stokes equations with the pressure function  $p \sim \varrho^\gamma$  for all  $\gamma > 5/3$  in  $3D$  case and for all  $\gamma > 3/2$  in  $2D$  case. More recently, Feireisl, Novotný, and Petzeltová, see [4], proved the existence result for all  $\gamma > 3/2$  in  $3D$  case and for all  $\gamma > 1$  in  $2D$  case, see also monographs [5], [7], and [10] for references and details. The question on solvability of problem (1) remained open. The main difficulty is the so called concentration problem, see [6] ch.6.6. This means that the finite kinetic energy can be concentrated in very small domains. Our goal is to relax the restriction  $\gamma > 1$  and to prove the existence of solutions to problem (1). In order to make the presentation clearer and avoid unnecessary technical difficulties, we assume that the flow domain and the given data satisfy the hypotheses:

**Condition 1.1.** • *The flow domain  $\Omega \subset \mathbb{R}^2$  is a bounded domain with  $C^\infty$  boundary.*

• *The data satisfy  $\varrho_0, \mathbf{u}_0 \in L^\infty(\Omega)$ ,  $\mathbf{f} \in L^\infty(Q_T)$ , and*

$$\|\mathbf{u}_0\|_{W_0^{1,2}(\Omega)} + \|\varrho_0\|_{L^\infty(\Omega)} + \|\mathbf{f}\|_{L^\infty(Q_T)} \leq c_e, \quad \varrho_0 > c > 0, \quad (2)$$

*where  $c_e, c$  are positive constants.*

*Remark 1.1.* Further, we denote by  $E$  generic constants depending only on  $\Omega, T, \|\varrho_0\|_{L^\infty(\Omega)}, \|\mathbf{u}_0\|_{L^2(\Omega)}, \|\mathbf{f}\|_{L^\infty(Q_T)}$ , and  $\nu_i$ .

We claim that problem (1) admits a weak solution which is defined as follows:

**Definition 1.1.** A couple

$$\varrho \in L^\infty(0, T; L^1(\Omega)), \quad \mathbf{u} \in L^2(0, T; W_0^{1,2}(\Omega))$$

is said to be a weak solution to problem (1) if  $(\varrho, \mathbf{u})$  satisfies

- The kinetic energy is bounded, i.e.,  $\varrho|\mathbf{u}|^2 \in L^\infty(0, T; L^1(\Omega))$ . The density function is non-negative  $\varrho \geq 0$ .
- The integral identity

$$\begin{aligned} \int_{Q_T} (\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\xi} + \varrho \operatorname{div} \boldsymbol{\xi} - \mathbb{S}(\mathbf{u}) : \nabla \boldsymbol{\xi}) dxdt \\ + \int_{Q_T} \varrho \mathbf{f} \cdot \boldsymbol{\xi} dxdt + \int_{\Omega} (\varrho_0 \mathbf{u}_0)(x) \cdot \boldsymbol{\xi}(x, 0) dx = 0 \end{aligned} \quad (3)$$

holds for all vector fields  $\boldsymbol{\xi} \in C^\infty(Q_T)$  vanishing in a neighborhood of  $\partial\Omega \times [0, T]$  and of  $\Omega \times \{t = T\}$ .

- The integral identity

$$\int_{Q_T} (\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi) dxdt + \int_{\Omega} \varrho_0(x) \psi(x, 0) dx = 0 \quad (4)$$

holds for all  $\psi \in C^\infty(Q_T)$  vanishing in a neighborhood of the top  $\Omega \times \{t = T\}$ .

The following existence theorem is the main result of the paper.

**Theorem 1.1.** Assume that Condition 1.1 is fulfilled. Then problem (1) has a weak solution which meets all requirements of Definition 1.1 and satisfies the estimate

$$\|\mathbf{u}\|_{L^2(0, T; W_0^{1,2}(\Omega))} + \|\varrho|\mathbf{u}|^2\|_{L^\infty(0, T; L^1(\Omega))} + \|\varrho \log(1 + \varrho)\|_{L^\infty(0, T; L^1(\Omega))} \leq E, \quad (5)$$

where the constant  $E$  is as in Remark 1.1.

The next theorem, which is the second main result of the paper, shows that a weak solution to problem (1) has extra regularity properties.

**Theorem 1.2.** *Let Condition 1.1 be satisfied. Assume that  $(\varrho, \mathbf{u})$  meets all requirements of Theorem 1.1. Furthermore assume that  $\mathbf{u}$  and  $\varrho$  are extended by 0 to  $\mathbb{R}^2 \times (0, T)$ . Then for every nonnegative function  $\zeta \in C_0^\infty(\mathbb{R}^2)$  with  $\text{spt } \zeta \Subset \Omega$ ,*

$$\text{ess sup}_{\omega \in \mathbb{S}^1} \int_0^T \int_{-\infty}^{\infty} \Phi(\omega, \tau, t)^2 d\tau dt \leq c(\zeta)E, \quad (6)$$

where  $\Phi$  is the Radon transform of  $\zeta(x)\varrho(x, t)$ ,

$$\Phi(\omega, \tau, t) = \int_{\omega \cdot x = \tau} \zeta(x)\varrho(x, t) dl. \quad (7)$$

Moreover, the function  $\zeta\varrho$  admits the estimates

$$\begin{aligned} \|\zeta\varrho\|_{L^2(0, T; H^{-1/2}(\mathbb{R}^2))} &\leq c(\zeta)E, \\ \|\zeta\varrho\|_{L^{1+\lambda}(Q_T)} &\leq c(\zeta, \lambda)E \quad \text{for all } \lambda \in [0, 1/6). \end{aligned} \quad (8)$$

Here  $c(\zeta)$  depends only on  $\zeta$  and  $c(\zeta, \lambda)$  depends only on  $\zeta, \lambda$ .

The remaining part of the paper is devoted to the proof of these theorems. In sections 2 and 3 we collect basic facts on Sobolev spaces, the Radon transform, and the isentropic Navier-Stokes equations. Section 4 is the heart of the work. Here we derive the  $L^2$ -estimates for the Radon transform of the density function  $\varrho$ . In sections 5 and 6 we prove that the density is locally integrable with exponent  $1 + \lambda < 7/6$ . In section 7 we complete the proof of Theorems 1.1 and 1.2.

## 2 Preliminaries

### 2.1 Sobolev spaces. Radon transform. Multipliers

For every  $s \in \mathbb{R}$ , denote by  $H^s(\mathbb{R}^2)$  the Sobolev space of all tempered distributions  $u$  in  $\mathbb{R}^2$  with the finite norm

$$\|u\|_{H^s(\mathbb{R}^2)} = \|(1 + |\xi|^2)^{s/2} \mathfrak{F}u\|_{L^2(\mathbb{R}^2)}, \quad (9)$$

where  $\mathfrak{F}u(\xi)$  is the Fourier transform of  $u$ . For all nonnegative integers  $k$ , the space  $H^k(\mathbb{R}^2)$  coincides with  $W^{k,2}(\mathbb{R}^2)$ . For every  $u \in L^2(\mathbb{R}^2)$  and  $s \geq 0$  we have

$$\|u\|_{H^{-s}(\mathbb{R}^2)} = \sup_{g \in H^s(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} ug \, dx}{\|g\|_{H^s(\mathbb{R}^2)}}. \quad (10)$$

Introduce the Bessel kernel  $G_1 = \mathfrak{F}^{-1}(1 + |\xi|^2)^{-1/2}$ . It is well-known that it is strictly positive and analytic in  $\mathbb{R}^2 \setminus \{0\}$ . Moreover, the Bessel kernel admits the estimates

$$c^{-1}|z|^{-1} \leq G_1(z) \leq c|z|^{-1} \quad \text{for } |z| \leq 1, \quad G_1(z) \leq c|z|^{-1}e^{-|z|} \quad \text{for } |z| \geq 1. \quad (11)$$

In particular, for every  $N > 0$  there exists a constant  $e(N) > 0$  with the property

$$e(N)|z|^{-1} \leq G_1(z) \leq c|z|^{-1} \quad \text{for } |z| \leq N. \quad (12)$$

The equality

$$\|G_1 * u\|_{H^{s+1}(\mathbb{R}^2)} = \|u\|_{H^s(\mathbb{R}^2)}. \quad (13)$$

holds true for all  $u \in H^s(\mathbb{R}^2)$ ,  $s \in \mathbb{R}$ .

The next lemma constitutes Sobolev estimates for the functions with integrable Radon transform.

**Lemma 2.1.** *Let  $g \in L^2(\mathbb{R}^2)$  be a compactly supported. Then*

$$\|g\|_{H^{-1/2}(\mathbb{R}^2)}^2 \leq \frac{1}{4\pi} \int_{\mathbb{S}^1 \times \mathbb{R}} \Phi(\omega, \tau)^2 \, d\omega \, d\tau, \quad \text{where } \Phi(\omega, \tau) = \int_{\omega \cdot x = \tau} g(x) \, dl. \quad (14)$$

*Proof.* The proof is in Appendix A □

The last lemma concerns multiplicative properties of Sobolev spaces.

**Lemma 2.2.** *Let  $s > 1/2$ ,  $g \in L^2(\mathbb{R}^2)$  and  $u \in H^1(\mathbb{R}^2)$ . Then there is  $c(s) > 0$  such that*

$$\|gu\|_{H^{-s}(\mathbb{R}^2)} \leq c(s)\|g\|_{H^{-1/2}(\mathbb{R}^2)}\|u\|_{H^1(\mathbb{R}^2)}. \quad (15)$$

*Proof.* The proof is in Appendix A □

## 2.2 Poisson equation

Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain and  $r \in (1, \infty)$ . Let  $f \in L^r(\mathbb{R}^2)$  be an arbitrary function such that  $\text{spt } f \subset \Omega$ . Then, see [3], the Poisson equation

$$\Delta u = f \quad \text{in } \mathbb{R}^2, \quad (16)$$

has a solution with the properties: This solution is analytic outside of  $\Omega$ , and satisfies

$$\limsup_{|x| \rightarrow \infty} (\log |x|)^{-1} |u(x)| < \infty, \quad \|u\|_{W^{1,2}(B_R)} \leq c \|f\|_{L^r(\mathbb{R}^d)}.$$

Here  $B_R$  is the ball  $\{x \in \mathbb{R}^d : |x| < R\}$  of an arbitrary radius  $R < \infty$ , and the constant  $c$  depends only on  $R$  and  $\Omega$ . The relation  $f \rightarrow u$  determines a linear operator  $\Delta^{-1}$ . In this framework we can define the linear operators

$$A_j = \partial_{x_j} \Delta^{-1}, \quad R_j = \partial_{x_j} (-\Delta)^{-1/2}, \quad j = 1, 2.$$

The Riesz operator  $R_j$  is a singular integral operator and by the Zygmund-Calderón theorem it is bounded in any space  $L^p(\mathbb{R}^d)$  with  $1 < p < \infty$ . In particular we have

$$\begin{aligned} \|A_j f\|_{W^{1,r}(B_R)} &\leq c(R, \Omega) \|f\|_{L^r(\mathbb{R}^2)} \quad \text{when } \text{spt } f \subset \Omega, \\ \|R_j f\|_{L^p(\mathbb{R}^2)} &\leq c(r) \|f\|_{L^r(\mathbb{R}^2)}. \end{aligned}$$

Notice that these operators have integral representations. In particular, we have

$$A_i f(x) = c \int_{\mathbb{R}^2} |x - y|^{-2} (x_i - y_i) f(y) dy. \quad (17)$$

## 3 Regularized problem

In order to regularize problem (1) we use the artificial pressure method and replace equations (1) by regularized equations

$$\partial_t(\varrho \mathbf{u}) + \text{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \text{div } \mathbb{S}(\mathbf{u}) + \varrho \mathbf{f} \quad \text{in } Q_T, \quad (18a)$$

$$\partial_t \varrho + \text{div}(\varrho \mathbf{u}) = 0 \quad \text{in } Q_T, \quad (18b)$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (18c)$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \varrho(x, 0) = \varrho_0(x) \quad \text{in } \Omega. \quad (18d)$$

Here, the artificial pressure function is given by

$$p(\varrho) = \varrho + \varepsilon \varrho^\gamma, \quad \varepsilon \in (0, 1], \quad \gamma \geq 6. \quad (18e)$$

The existence of weak renormalized solutions to problem (18) was established in monographs [5] and [6]. The following proposition is a consequence of these results.

**Proposition 3.1.** *Let domain  $\Omega$ , and functions  $\mathbf{u}_0$ ,  $\varrho_0$ ,  $\mathbf{f}$  satisfy Condition 1.1. Then problem (18) has a weak solution  $(\varrho, \mathbf{u})$  with the following properties:*

(i) *The functions  $\varrho \geq 0$  and  $\mathbf{u}$  satisfy the energy inequality*

$$\operatorname{ess\,sup}_{t \in (0, T)} \int_{\Omega} \{ \varrho |\mathbf{u}|^2 + \varrho \ln(1 + \varrho) + \varepsilon \varrho^\gamma \} (x, t) dx + \int_{Q_T} |\nabla \mathbf{u}|^2 dx dt \leq cE. \quad (19)$$

*The constant  $E$  is as in Remark 1.1.*

(ii) *The integral identity*

$$\begin{aligned} \int_{Q_T} (\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi}) dx dt + \int_{Q_T} (\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} - \mathbb{S}(\mathbf{u})) : \nabla \boldsymbol{\xi} dx dt + \\ \int_{Q_T} \varrho \mathbf{f} \cdot \boldsymbol{\xi} dx dt + \int_{\Omega} \varrho_0(x) \mathbf{u}_0(x) \cdot \boldsymbol{\xi}(x, 0) dx = 0 \end{aligned} \quad (20)$$

*holds for all vector fields  $\boldsymbol{\xi} \in C^\infty(Q)$  satisfying*

$$\boldsymbol{\xi}(x, T) = 0 \quad \text{in } \Omega, \quad \boldsymbol{\xi}(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T). \quad (21)$$

(iii) *The integral identity*

$$\begin{aligned} \int_{Q_T} \left( \varphi(\varrho) \partial_t \psi + (\varphi(\varrho) \mathbf{u}) \cdot \nabla \psi - \psi (\varphi'(\varrho) \varrho - \varphi(\varrho)) \operatorname{div} \mathbf{u} \right) dx dt \\ + \int_{\Omega} (\psi \varphi(\varrho_0))(x, 0) dx = 0 \end{aligned} \quad (22)$$

*holds for all smooth functions  $\psi$ , vanishing in a neighborhood of the top  $\Omega \times \{t = T\}$ , and for all functions  $\varphi \in C^2[0, \infty)$  satisfying the growth condition*

$$|\varphi(\varrho)| + |\varphi'(\varrho) \varrho| + |\varphi''(\varrho) \varrho^2| \leq C(1 + \varrho^2). \quad (23)$$

*Remark 3.1.* Further we will assume that  $(\varrho, \mathbf{u})$  and  $(\varrho_0, \mathbf{u}_0)$  are extended by 0 to the layer

$$\Pi = \mathbb{R}^2 \times (0, T). \quad (24)$$

The following consequences of Proposition 3.1 will be used throughout the paper.

**Corollary 3.1.** *Assume that  $(\varrho, \mathbf{u})$  meets all requirements of Proposition 3.1. Then there is a constant  $c(E, \varepsilon)$ , depending only on  $E$ ,  $\beta$ ,  $\alpha$ ,  $\gamma$ , and  $\varepsilon$ , such that*

$$\begin{aligned} \|\varrho \mathbf{u}\|_{L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega))} &\leq c(E, \varepsilon), \\ \|\varrho \mathbf{u}\|_{L^2(0, T; L^\beta(\Omega))} &\leq c(E, \varepsilon) \quad \text{for all } \beta \in [1, \infty] \quad \text{with } \beta < \gamma, \\ \|\varrho \mathbf{u}\|_{L^\alpha(0, T; L^2(\Omega))} &\leq c(E, \varepsilon) \quad \text{for all } \alpha \in [1, 2\gamma - 2). \end{aligned}$$

*Proof.* It follows from (19) that

$$\begin{aligned} \|\varrho\|_{L^\infty(0, T; L^\gamma(\Omega))} + \|\varrho |\mathbf{u}|^2\|_{L^\infty(0, T; L^1(\Omega))} &\leq c(E, \varepsilon), \\ \|\mathbf{u}\|_{L^2(0, T; W_0^{1,2}(\Omega))} &\leq c(E, \varepsilon). \end{aligned} \quad (25)$$

Hence  $(\varrho, \mathbf{u})$  are bounded energy functions and the corollary is a particular case of Corollary 4.2.2 in [10].  $\square$

**Corollary 3.2.** *Assume that  $(\varrho, \mathbf{u})$  meets all requirements of Proposition 3.1. Then there is a constant  $c(E, \varepsilon)$ , depending only on  $E$ ,  $\gamma$ , and  $\varepsilon$ , such that*

$$\|\varrho |\mathbf{u}|^2\|_{L^2(0, T; L^\tau(\Omega))} \leq c(E, \varepsilon) \quad \text{for all } \tau \in [1, 2\gamma/(\gamma+1)), \quad (26)$$

$$\|\varrho |\mathbf{u}|^2\|_{L^1(0, T; L^\tau(\Omega))} \leq c(E, \varepsilon) \quad \text{for all } \tau \in [1, \gamma). \quad (27)$$

*Proof.* In view of (25) the corollary is a particular case of Corollary 4.2.3 in [10].  $\square$

## 4 Radon transform

In this section we estimate the Radon transform of solutions to regularized equations (18). The corresponding result is given by the following theorem. Fix an arbitrary function  $\zeta$  with the properties

$$\zeta \in C_0^\infty(\mathbb{R}^2), \quad \text{spt } \zeta \Subset \Omega, \quad \zeta \geq 0. \quad (28)$$



**Theorem 4.1.** *Assume that a renormalized solution to problem (18) meets all requirements of Proposition 3.1. Furthermore, assume that  $\mathbf{u}$  and  $\varrho$  are extended by 0 to the layer  $\Pi$ . Then for every unit vector  $\boldsymbol{\omega} \in \mathbb{R}^2$ ,*

$$\int_0^T \int_{-\infty}^{\infty} \left\{ \int_{\boldsymbol{\omega} \cdot \mathbf{x} = \tau} \zeta(x) \varrho(x, t) dl \right\}^2 d\tau dt \leq c(\zeta) E, \quad (29)$$

where  $c(\zeta)$  depends only on  $\zeta$ , and  $E$  is specified by Remark 1.1. Notice that  $c(\zeta)$  and  $E$  are independent of  $\boldsymbol{\omega}$  and  $\varepsilon$ .

Since the Navier-Stokes equations are invariant with respect to rotations, it suffices to prove (29) for  $\boldsymbol{\omega} = (1, 0)$ , i.e., to prove the inequality

$$\int_0^T \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \zeta(x) \varrho(x, t) dx_2 \right\}^2 dx_1 dt \leq c(\zeta) E. \quad (30)$$

We split the proof of (30) into a sequence of lemmas.

**Lemma 4.1.** *Let all hypotheses of Theorem 4.1 be satisfied. Then for every function  $\varphi \in C^\infty(Q_T)$  vanishing in a neighborhood of  $\partial\Omega \times (0, T)$ ,*

$$\begin{aligned} \int_{Q_T} \left( \varrho u_1 \cdot \partial_t \varphi + (\varrho u_1 u_i - \mathbb{S}(\mathbf{u})_{i1}) \frac{\partial \varphi}{\partial x_i} + p \frac{\partial \varphi}{\partial x_1} \right) dx dt + \\ \int_{Q_T} \varrho f_1 \varphi dx dt \leq c(\zeta) E \|\varphi\|_{L^\infty(Q_T)}. \end{aligned} \quad (31)$$

*Proof.* Set

$$\eta_h(t) = 1 \quad \text{for } t \leq T - h, \quad \eta = \frac{1}{h}(T - t) \quad \text{for } t \in [T - h, T]. \quad (32)$$

Substituting  $\boldsymbol{\xi} = (\eta_h \varphi, 0)$  into (20) we arrive at the identity

$$\begin{aligned} \int_{Q_T} \eta_h \varrho u_1 \cdot \partial_t \varphi dx dt + \int_{Q_T} \eta_h \left( (\varrho u_1 u_i - \mathbb{S}(\mathbf{u})_{i1}) \frac{\partial \varphi}{\partial x_i} + p \frac{\partial \varphi}{\partial x_1} \right) dx dt + \\ \int_{Q_T} \eta_h \varrho f_1 \varphi dx dt = \frac{1}{h} \int_{T-h}^T \int_{\Omega} \varrho \mathbf{u}_1 \varphi dx dt - \int_{\Omega} \varrho_0 \mathbf{u}_{1,0} \varphi(x, 0) dx. \end{aligned} \quad (33)$$

Next notice that

$$\begin{aligned} \left| \frac{1}{h} \int_{T-h}^T \int_{\Omega} \varrho \mathbf{u}_1 \varphi dx dt \right| + \left| \int_{\Omega} \varrho_0 \mathbf{u}_{1,0} \varphi(x, 0) dx \right| \leq \\ \|\varrho \mathbf{u}\|_{L^\infty(0, T; L^1(\Omega))} \|\varphi\|_{L^\infty(Q_T)} + \|\varrho_0 \mathbf{u}_0\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(Q_T)} \leq E \|\varphi\|_{L^\infty(Q_T)}. \end{aligned}$$

Letting  $h \rightarrow 0$  in (33) we arrive at (31)  $\square$

Now we specify the test function  $\varphi$ . Choose  $\omega : \mathbb{R} \rightarrow \mathbb{R}^+$  satisfying

$$\omega \in C_0^\infty(\mathbb{R}), \quad \text{spt } \omega \subset [-1, 1], \quad \omega \text{ is even}, \quad \int_{\mathbb{R}} \omega(s) ds = 1,$$

For every  $f \in L_{\text{loc}}^1(\mathbb{R}^2)$  define the mollifiers

$$[f]_h = \frac{1}{h^2} \int_{\mathbb{R}^2} \omega\left(\frac{x_1 - y_1}{h}\right) \omega\left(\frac{x_2 - y_2}{h}\right) f(y) dy. \quad (34)$$

Introduce the auxiliary functions

$$H(x_1, t) = \int_{-\infty}^{x_1} \Psi(s, t) ds, \quad \Psi(x_1, t) = \int_{\mathbb{R}} [\zeta \varrho]_h(x_1, x_2, t) dx_2, \quad (35a)$$

and take the test function  $\varphi$  in the form

$$\varphi(x, t) = \zeta(x) [H]_h(x_1, t). \quad (35b)$$

The following lemma constitutes properties of  $\Psi$  and  $H$ .

**Lemma 4.2.**  $\Psi, H \in L^\infty(0, T; C^k(\mathbb{R}))$  for every integer  $k \geq 0$ , and

$$\|H\|_{L^\infty(\mathbb{R} \times (0, T))} \leq c(\zeta)E. \quad (36)$$

*Proof.* Notice that  $\zeta \varrho \in L^\infty(0, T; L^1(\mathbb{R}^2))$  and its norm in this space does not exceed  $E$ . Hence for a.e.  $t \in (0, T)$ ,

$$\|[\zeta \varrho]_h(t)\|_{C^k(\mathbb{R}^2)} \leq c(k)\|\zeta \varrho(t)\|_{L^1(\mathbb{R}^2)} \leq c(k)E.$$

Hence  $[\zeta \varrho]_h$  belongs to  $L^\infty(0, T; C^k(\mathbb{R}))$ . Next, there is  $N$  such that the square  $[-N + 1, N - 1]^2$  contains domain  $\Omega$ . Hence the function  $[\zeta \varrho]_h(t)$  is compactly supported in the square  $[-N, N]^2$ . It follows that  $\Psi \in L^\infty(0, T; C^k(\mathbb{R}))$  and  $\Psi(\cdot, t)$  is supported in the interval  $[-N, N]$ . From this and (35) we conclude that  $H \in L^\infty(0, T; C^k(\mathbb{R}))$ . It remains to note that

$$|H(x_1, t)| \leq \int_{\mathbb{R}} \Psi(x_1, t) dx_1 \leq \int_{\mathbb{R}^2} [\zeta \varrho]_h dx dt = \int_{\mathbb{R}^2} \zeta \varrho dx dt = \int_{\Omega} \zeta(x) \varrho(x, t) dx \leq cE.$$

□

Now we investigate in details the time dependence of  $H$ .

**Lemma 4.3.** *The function  $\partial_t H$  belongs to the class  $L^\infty(0, T; C^k(\Omega))$  for every integer  $k \geq 0$ . Moreover, it has the representation*

$$\partial_t H = -v_h + J_0, \quad \text{where } v_h(x_1, t) = \int_{\mathbb{R}} [\zeta \varrho u_1]_h(x_1, x_2, t) dx_2, \quad (37)$$

and the reminder admits the estimate

$$|J_0| \leq c(\zeta)E. \quad (38)$$

*Proof.* Integral identity (22) with  $\varphi(\varrho) = \varrho$  and  $\psi$  replaced by  $\zeta\psi$  reads

$$\int_{\Pi} \left( \zeta \varrho \partial_t \psi + \zeta \varrho \mathbf{u} \cdot \nabla \psi + \psi \varrho \nabla \zeta \mathbf{u} \right) dx dt + \int_{\mathbb{R}^2} \psi(x, 0) \zeta(x) \varrho_0(x) dx = 0. \quad (39)$$

This identity holds true for all functions  $\psi \in C^\infty(\mathbb{R}^2 \times (0, T))$ , vanishing in a neighborhood of the top  $\mathbb{R}^2 \times \{t = T\}$ . Now choose an arbitrary  $\xi \in C_0^\infty(0, T)$  and  $y \in \mathbb{R}^2$ . Inserting

$$\psi = \xi(t) h^{-2} \omega\left(\frac{x_1 - y_1}{h}\right) \omega\left(\frac{x_2 - y_2}{h}\right)$$

into (39) we arrive at

$$\int_0^T \left( [\zeta \varrho]_h(y, t) \xi'(t) - \xi \operatorname{div} [\zeta \varrho \mathbf{u}]_h(y, t) + \xi [\varrho \nabla \zeta]_h(y, t) \right) dt = 0,$$

which yields

$$\partial_t [\zeta \varrho]_h = -\operatorname{div} [\zeta \varrho \mathbf{u}]_h + [\varrho \nabla \zeta \cdot \mathbf{u}]_h \quad \text{in } \mathbb{R}^2 \times [0, T]. \quad (40)$$

Next, Corollary 3.1 implies that  $\zeta \varrho \mathbf{u}$  and  $\varrho \nabla \zeta \cdot \mathbf{u}$  belong to  $L^\infty(0, T; L^1(\mathbb{R}^2))$ . Hence the functions  $\operatorname{div} [\zeta \varrho \mathbf{u}]_h$  and  $[\varrho \nabla \zeta \cdot \mathbf{u}]_h$  belong to  $L^\infty(0, T; C^k(\mathbb{R}^2))$  for all integer  $k \geq 0$ . Moreover, they are supported in  $Q_T$ . It follows that  $\partial_t [\zeta \varrho]_h$  belongs to  $L^\infty(0, T; C^k(\mathbb{R}^2))$  and is supported in  $Q_T$ . Therefore, the function

$$\partial_t H = \int_{-\infty}^{x_1} \int_{-\infty}^{\infty} \partial_t [\zeta \varrho]_h(s, x_2, t) ds dx_2$$

belongs to the class  $L^\infty(0, T; C^k(\mathbb{R}^2))$ . Integrating both sides of (40) over  $(-\infty, x_1] \times \mathbb{R}$  we obtain representation (37) with the reminder

$$J_0(x_1, t) = \int_{-\infty}^{x_1} \left\{ \int_{\mathbb{R}} [\varrho \nabla \zeta \cdot \mathbf{u}]_h(s, x_2, t) dx_2 \right\} ds.$$

It remains to note that for a. e.  $t \in (0, T)$ , we have

$$\begin{aligned} |J_0(x_1, t)| &\leq \int_{\mathbb{R}^2} [\varrho |\nabla \zeta| |\mathbf{u}|]_h(x, t) dx = \int_{\mathbb{R}^2} \varrho |\nabla \zeta| |\mathbf{u}|(x, t) dx \leq \\ &c(\zeta) \int_{\Omega} \varrho |\mathbf{u}|(x, t) dx \leq c(\zeta) \|\varrho \mathbf{u}\|_{L^\infty(0, T; L^1(\Omega))} \leq c(\zeta) E. \end{aligned}$$

□

In view of Lemma 4.3 the function  $\varphi$  given by formula (35) belongs to  $L^\infty(0, T; C^k(\mathbb{R}^2))$  and is supported in  $Q_T$ . Moreover, we have

$$\|\varphi\|_{L^\infty(Q_T)} \leq c(\zeta) \|H\|_{L^\infty(\mathbb{R} \times (0, T))} \leq c(\zeta) E. \quad (41)$$

Substituting  $\varphi$  in (31) and using (41) we obtain

$$I_1 + I_2 + I_3 + I_4 + I_5 \leq c(\zeta) E \quad (42)$$

where

$$\begin{aligned} I_1 &= \int_{\Pi} \varrho u_1 \partial_t \varphi dx dt, \quad I_2 = \int_{\Pi} \varrho u_1 u_i \partial_{x_i} \varphi dx dt, \quad I_3 = \int_{\Pi} p \partial_{x_1} \varphi dx dt, \\ I_4 &= - \int_{\Pi} \mathbb{S}_{i1} \partial_{x_i} \varphi dx dt, \quad I_5 = \int_{\Pi} \varrho f_1 \varphi dx dt. \end{aligned} \quad (43)$$

Let us consider each term in (42) separately.

**Lemma 4.4.**

$$I_1 = - \int_0^T \int_{\mathbb{R}} v_h^2 dx_1 dt + J_1, \quad \text{where } |J_1| \leq c(\zeta) E. \quad (44)$$

*Proof.* Since  $\varrho u_1 \in L^2(\Pi)$  and the mollifying operator is symmetric, it follows from (35) that

$$I_1 = \int_{\Pi} \left[ \zeta \varrho u_1 \right]_h(x_1, x_2, t) \partial_t H(x_1, t) dx dt.$$

Next, the function  $\left[ \zeta \varrho u_1 \right]_h$  is supported in  $\Omega \times [0, T]$ . Therefore, the function  $v_h$  is supported in every rectangular  $[-N, N] \times [0, T]$  such that  $[-N, N]^2 \supset \Omega$ . From this we conclude that

$$I_1 = \int_0^T \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \left[ \zeta \varrho u_1 \right]_h dx_2 \right\} \partial_t H(x_1, t) dx_1 dt = \int_0^T \int_{\mathbb{R}} v_h(x_1, t) \partial_t H(x_1, t) dx_1 dt.$$

Inserting expression (37) for  $\partial_t H$  we obtain representation (44) with the reminder

$$J_1 = \int_0^T \int_{\mathbb{R}} J_0 v_h dx_1 dt.$$

It remains to note that in view of (38),

$$\begin{aligned} |J_1| &\leq c(\zeta)E \int_0^T \int_{\mathbb{R}} |v_h| dx_1 dt \leq c(\zeta)E \int_{\Pi} [\zeta \varrho |\mathbf{u}|]_h dx dt \\ &= cE \int_{\Pi} \zeta \varrho |\mathbf{u}| dx dt = cE \int_{Q_T} \zeta \varrho |\mathbf{u}| dx dt \leq c(\zeta)E \int_{Q_T} \varrho |\mathbf{u}| dx dt \leq c(\zeta)E. \end{aligned}$$

□

**Lemma 4.5.**

$$I_2 = \int_0^T \int_{\mathbb{R}} \Upsilon_h(x_1, t) \Psi(x_1, t) dx_1 dt + J_2, \text{ where } \Upsilon_h = \int_{\mathbb{R}} [\zeta \varrho u_1^2]_h(x_1, x_2, t) dx_2, \quad (45)$$

and the reminder  $J_2$  admits the estimate

$$|J_2| \leq c(\zeta)E. \quad (46)$$

Moreover, the function  $\Upsilon_h$  belongs to the class  $L^\infty(0, T; C^k(\mathbb{R}))$  for every integer  $k \geq 0$ . It is supported in any rectangular  $[-N, N] \times [0, T]$  such that  $[-N, N]^2 \supset \Omega$ .

*Proof.* Notice that  $\zeta \varrho u_i u_1 \in L^\infty(0, T; L^1(\Omega))$  is supported in  $Q_T$ . It follows from (35) that

$$I_2 = \int_{\Pi} \zeta \varrho u_1^2 \left[ \frac{\partial H}{\partial x_1} \right]_h dx dt + J_2, \text{ where } J_2 = \int_{\Pi} \varrho u_1 (\nabla \zeta \cdot \mathbf{u}) [H]_h dx dt.$$

Since  $\partial_{x_1} H = \Psi$  is independent of  $x_2$ , we have

$$\begin{aligned} \int_{\Pi} \zeta \varrho u_1^2 \left[ \frac{\partial H}{\partial x_1} \right]_h dx dt &= \int_{\Pi} \zeta \varrho u_1^2 [\Psi]_h dx dt = \int_{\Pi} [\zeta \varrho u_1^2]_h \Psi dx dt = \\ &= \int_0^T \int_{\mathbb{R}} \Psi(x_1, t) \left\{ \int_{\mathbb{R}} [\zeta \varrho u_1^2]_h dx_2 \right\} dx_1 dt = \int_0^T \int_{\mathbb{R}} \Upsilon_h(x_1, t) \Psi(x_1, t) dx_1 dt. \end{aligned}$$

This leads to the desired representation (45). In order to estimate  $J_2$  notice that in view of (36), we have  $\sup |[H]_h| \leq \sup H \leq c(\zeta)E$ . This gives

$$|J_2| \leq c(\zeta)E \int_{Q_T} \varrho |\mathbf{u}|^2 dx dt \leq c(\zeta)E.$$

□

**Lemma 4.6.**

$$I_3 \geq \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt + J_3, \quad \text{where } |J_3| \leq c(\zeta, E). \quad (47)$$

*Proof.* We have

$$\partial_{x_1} \varphi = \partial_{x_1} \zeta [H]_h + \zeta [\Psi]_h.$$

Hence

$$I_3 = \int_{\Pi} \zeta p [\Psi]_h dx dt + J_3, \quad \text{where } J_3 = \int_{\Pi} \partial_{x_1} \zeta p [H]_h dx dt.$$

Since  $\Psi$  is nonnegative we have

$$\begin{aligned} \int_{\Pi} \zeta p [\Psi]_h dx dt &= \int_{\Pi} \zeta \varrho [\Psi]_h dx dt + \varepsilon \int_{\Pi} \zeta \varrho^\gamma [\Psi]_h dx dt \geq \\ &\int_{\Pi} \zeta \varrho [\Psi]_h dx dt = \int_{\Pi} [\zeta \varrho]_h \Psi dx dt = \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt, \end{aligned}$$

which leads to (47). It remains to estimate  $J_3$ . To this end notice that in view of (36) and (19),

$$|J_3| \leq c(\zeta) \|H\|_{L^\infty(\mathbb{R} \times (0, T))} \int_0^T \int_{\Omega} p dx dt \leq c(\zeta)E.$$

□

**Lemma 4.7.**

$$|I_4| \leq c(\zeta)E + c(\zeta)E \left( \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt \right)^{1/2}, \quad |I_5| \leq c(\zeta)E. \quad (48)$$

*Proof.* It follows from formulae (35) that

$$I_4 = \int_{\Pi} \zeta \mathbb{S}_{11}(\mathbf{u}) [\Psi]_h dxdt + \int_{\Pi} (\partial_{x_i} \zeta) \mathbb{S}_{i1}(\mathbf{u}) [H]_h dxdt. \quad (49)$$

Notice that  $[\zeta \mathbb{S}]_h$  is compactly supported in  $\Omega \times [0, T]$ . Hence it is supported in the slab  $[-N, N]^2 \times [0, T]$  such that  $[-N, N]^2 \supset \Omega$ . Thus we get

$$\begin{aligned} \left| \int_{\Pi} \zeta \mathbb{S}_{11} [\Psi]_h dxdt \right| &= \left| \int_{\Pi} [\zeta \mathbb{S}_{11}]_h \Psi dxdt \right| = \left| \int_0^T \int_{[-N, N]^2} [\zeta \mathbb{S}_{11}]_h \Psi dxdt \right| \leq \\ &\left( \int_0^T \int_{[-N, N]^2} [\zeta \mathbb{S}_{11}]_h^2 dxdt \right)^{1/2} \left( \int_0^T \int_{[-N, N]^2} \Psi^2 dxdt \right)^{1/2} \leq \\ &\left( \int_0^T \int_{[-N, N]^2} (\zeta \mathbb{S}_{11})^2 dxdt \right)^{1/2} \left( N \int_0^T \int_{[-N, N]} \Psi^2 dx_1 dt \right)^{1/2} \leq \\ &c(\zeta) N^{1/2} \left( \int_{Q_T} |\nabla \mathbf{u}|^2 dxdt \right)^{1/2} \left( \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt \right)^{1/2} \end{aligned}$$

Since  $N$  depends only on  $\Omega$ , these inequalities along with estimate (19) imply

$$\left| \int_{\Pi} \zeta \mathbb{S}_{11} [\Psi]_h dxdt \right| \leq c(\zeta) E \left( \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt \right)^{1/2} \quad (50)$$

We also have

$$\left| \int_{\Pi} (\partial_{x_i} \zeta) \mathbb{S}_{i1}(\mathbf{u}) [H]_h dxdt \right| \leq c(\zeta) \|H\|_{L^\infty(\mathbb{R} \times (0, T))} \int_{Q_T} |\nabla \mathbf{u}| dxdt \leq c(\zeta) E. \quad (51)$$

Inserting (50) and (51) into (49) we arrive at the first inequality in (48). It remains to notice that

$$|I_5| \leq c \|f\|_{L^\infty(Q_T)} \|H\|_{L^\infty(\mathbb{R} \times (0, T))} \int_{\Pi} \zeta \varrho dxdt \leq c(\zeta) E \int_{Q_T} \varrho dxdt \leq c(\zeta) E.$$

□

**Lemma 4.8.**

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt + \int_0^T \int_{\mathbb{R}} (\Upsilon_h \Psi - v_h^2) dx_1 dt \leq c(\zeta) E. \quad (52)$$

*Proof.* Estimate (48) for  $I_5$  and inequality (42) imply

$$I_1 + I_2 + I_3 + I_4 \leq c(\zeta)E.$$

From this and Lemmas 4.4, 4.5 we obtain

$$I_3 + \int_0^T \int_{\mathbb{R}} (\Upsilon_h \Psi - v_h^2) dx_1 dt \leq c(\zeta)E + |I_4|$$

Applying Lemmas 4.6 and 4.7 we arrive at

$$\int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt + \int_0^T \int_{\mathbb{R}} (\Upsilon_h \Psi - v_h^2) dx_1 dt \leq c(\zeta)E + c(\zeta)E \left( \int_0^T \int_{\mathbb{R}} \Psi^2 dx_1 dt \right)^{1/2},$$

which obviously leads to (52).  $\square$

**Lemma 4.9.**  $\Upsilon_h \Psi - v_h^2 \geq 0$  in  $\Pi$ .

*Proof.* We begin with the observation that the inequality

$$[fg]_h^2 \leq [f^2]_h [g^2]_h$$

holds for all functions  $f(x)$ ,  $g(x)$  locally integrable with square in  $\mathbb{R}^2$ . Setting  $f = \sqrt{\zeta \varrho}(\cdot, t)$  and  $g = \sqrt{\zeta \varrho} |u_1|(\cdot, t)$  we obtain for a.e.  $x, t$  and all  $\delta > 0$ ,

$$\begin{aligned} [\zeta \varrho |u_1|]_h(x_1, x_2, t) &\leq \sqrt{[\zeta \varrho u_1^2]_h(x_1, x_2, t)} \sqrt{[\zeta \varrho]_h(x_1, x_2, t)} \leq \\ &\frac{1}{2} \delta [\zeta \varrho u_1^2]_h(x_1, x_2, t) + \frac{1}{2} \delta^{-1} [\zeta \varrho]_h(x_1, x_2, t) \end{aligned}$$

Integrating both sides with respect to  $x_2$  over  $\mathbb{R}$  and recalling formulae (37) and (45) for  $v_h$  and  $\Upsilon_h$  we arrive at

$$v_h(x_1, t) \leq \frac{1}{2} \delta \Upsilon_h(x_1, t) + \frac{1}{2} \delta^{-1} \Psi(x_1, t).$$

Recall that  $\Upsilon_h$  and  $\Psi$  are nonnegative. Setting  $\delta = (\Psi/\Upsilon_h)^{1/2}$  we obtain the desired inequality.  $\square$

We are now in a position to complete the proof of Theorem 4.1. Introduce the function

$$\Phi_1(x_1, t) = \int_{\mathbb{R}} \zeta \varrho(x_1, x_2, t) dx_2 \equiv \Phi(\mathbf{e}_1, x_1, t).$$



Recall that  $\zeta \varrho$  is supported in  $Q_T$ . It suffices to prove that

$$\int_0^T \int_{\mathbb{R}} \Phi_1^2 dx_1 dt \leq c(\zeta)E. \quad (53)$$

By virtue of the energy estimate (19), the function  $\Phi_1$  belongs to the class  $L^2(0, T; \mathbb{R})$ . It is supported in the rectangular  $[-N, N] \times [0, T]$  for every  $N$  such that  $[-N, N]^2 \supset \Omega$ . It obviously follows from this and definition (34) of the mollifier that

$$\Psi = [\Phi_1]_h^{(1)}, \quad \text{where} \quad [\Phi_1]_h^{(1)}(x_1, t) = \frac{1}{h} \int_{\mathbb{R}} \omega\left(\frac{x_1 - y_1}{h}\right) \Phi_1(y_1, t) dy_1.$$

In other words,  $[\Phi_1]_h^{(1)}$  is the mollifying of  $\Phi_1$  with respect to  $x_1$ . Lemmas 4.8 and 4.9 imply the inequality

$$\int_0^T \int_{\mathbb{R}} ([\Phi_1]_h^{(1)})^2 dx_1 dt \leq c(\zeta)E. \quad (54)$$

Notice that  $[\Phi_1]_h^{(1)} \rightarrow \Phi_1$  a.e. in  $\mathbb{R} \times (0, T)$ . Letting  $h \rightarrow 0$  in (54) and applying the Fatou Theorem we arrive at (53).  $\square$

## 5 Momentum estimates

In this section we prove auxiliary estimates for solutions  $(\varrho, \mathbf{u})$  to regularized equations (18). We start with the estimating of norms of  $\varrho$  and  $\varrho \mathbf{u}$  in negative Sobolev spaces.

**Proposition 5.1.** *Let a solution  $(\varrho, \mathbf{u})$  to problem (18) meets all requirements of Proposition 3.1. Let  $\zeta \in C_0^\infty(\mathbb{R}^2)$  be an arbitrary nonnegative compactly supported in  $\Omega$  function and  $s > 1/2$ . Then*

$$\|\zeta \varrho\|_{L^2(0, T; H^{-1/2}(\mathbb{R}^2))} \leq c(\zeta)E, \quad (55)$$

$$\|\zeta \varrho \mathbf{u}\|_{L^1(0, T; H^{-s}(\mathbb{R}^2))} \leq c(\zeta)c(s)E, \quad (56)$$

where  $c(\zeta)$  depends only on  $\zeta$ ,  $c(s)$  depends only on  $s$ , and  $E$  is specified by Remark 1.1.

*Proof.* Lemma 2.1 and Theorem 4.1 imply the estimates

$$\begin{aligned} \int_0^T \|\zeta \varrho\|_{H^{-1/2}(\mathbb{R}^2)}^2 dt &\leq \int_0^T \int_{\mathbb{S}^1} \int_{-\infty}^{\infty} \left\{ \int_{\boldsymbol{\omega} \cdot \mathbf{x} = \tau} \zeta \varrho dl \right\}^2 d\tau d\boldsymbol{\omega} dt = \\ \int_{\mathbb{S}^1} \left\{ \int_0^T \int_{-\infty}^{\infty} \left\{ \int_{\boldsymbol{\omega} \cdot \mathbf{x} = \tau} \zeta \varrho dl \right\}^2 d\tau dt \right\} d\boldsymbol{\omega} &\leq c(\zeta) E \int_{\mathbb{S}^1} d\boldsymbol{\omega} \leq c(\zeta) E, \end{aligned}$$

which yield (55). Next, Lemma 2.2 implies the inequality

$$\|\zeta \varrho(t) \mathbf{u}(t)\|_{H^{-s}(\mathbb{R}^2)} \leq c(\zeta, s) \|\zeta \varrho(t)\|_{H^{-1/2}(\mathbb{R}^2)} \|\mathbf{u}(t)\|_{H^1(\mathbb{R}^2)}.$$

From this, (55), and estimate (19) we finally obtain

$$\|\zeta \varrho \mathbf{u}\|_{L^1(0,T;H^{-s}(\mathbb{R}^2))} \leq c(\zeta, s) \|\zeta \varrho\|_{L^2(0,T;H^{-1/2}(\mathbb{R}^2))} \|\mathbf{u}\|_{L^2(0,T;H^1(\mathbb{R}^2))} \leq c(\zeta, s) E.$$

□

## 5.1 Cauchy-Riemann equations

Further notation  $\nabla^\perp$  and  $\text{rot}$  stands for the differential operators

$$\nabla^\perp f = (-\partial_{x_2} f, \partial_{x_1} f), \quad \text{rot } \mathbf{w} = \partial_{x_2} w_1 - \partial_{x_1} w_2.$$

Denote by  $\mathbf{F} = (F_1, F_2)$  a solution to the inhomogeneous Cauchy-Riemann equation

$$\nabla F_1 + \nabla^\perp F_2 = \zeta \varrho \mathbf{u} \quad \text{in } \Pi. \quad (57)$$

It is easily seen that

$$F_1 = \text{div } \Delta^{-1}(\zeta \varrho \mathbf{u}), \quad F_2 = -\text{rot } \Delta^{-1}(\zeta \varrho \mathbf{u}). \quad (58)$$

The following two auxiliary lemmas give  $L^p$ - estimates for a vector function  $\mathbf{F}$ .

**Lemma 5.1.** *Under the assumptions of Proposition 5.1, for every positive  $\delta$  and  $R$ , there is a constant  $c(\delta, \zeta, R)$  such that*

$$\|\mathbf{F}\|_{L^4(0,T;L^{\frac{8}{3+\delta}}(B_R))} \leq c(\delta, \zeta, R) E. \quad (59)$$

*Proof.* Fix an arbitrary positive  $\delta$  and  $R$ . Without loss of generality we may assume that  $\delta < 1$  and  $B_R \supset \Omega$ . Integral representation (17) for the operator  $\partial_x \Delta^{-1}$  and formulae (58) for solutions to inhomogeneous Cauchy-Riemann equations imply the inequalities

$$|\mathbf{F}(x, t)| \leq c \int_{\Omega} |x - y|^{-1} \zeta \varrho |\mathbf{u}|(y, t) dy \leq \frac{c}{2} b(x, t) \mathcal{L}(x, t) + \frac{c}{2} b^{-1}(x, t) \mathcal{Q}(x, t),$$

where

$$\mathcal{L}(x, t) = \int_{\Omega} |x - y|^{-1} \zeta \varrho |\mathbf{u}|^2(y, t) dy, \quad \mathcal{Q}(x, t) = \int_{\Omega} |x - y|^{-1} \zeta \varrho dy,$$

$b$  is an arbitrary positive function. Notice that if  $\mathcal{L}(x, t)$  or  $\mathcal{Q}(x, t)$  vanishes at least at one point  $(x, t)$ , then  $\mathbf{F}(\cdot, t)$  vanishes in  $\mathbb{R}^2$ . In opposite case we can take  $b = \sqrt{\mathcal{Q}/\mathcal{L}}$ . Thus we get

$$|\mathbf{F}(x, t)| \leq c \sqrt{\mathcal{L}(x, t)} \sqrt{\mathcal{Q}(x, t)} \quad \text{a.e. in } \mathbb{R}^2 \times (0, T) \quad (60)$$

Now our task is to estimate  $\mathcal{L}$  and  $\mathcal{Q}$ . We have

$$\mathcal{L}(x, t) = \int_{B_R} (\zeta \varrho |\mathbf{u}|^2)^{\frac{1+\delta}{2}} |x - y|^{-1-\delta+\alpha} (\zeta \varrho |\mathbf{u}|^2)^{\frac{1-\delta}{2}} |x - y|^{\alpha} dy,$$

where  $\alpha = \delta/2 > 0$ . It follows that

$$\mathcal{L}(x, t) \leq c(R) \int_{B_R} (\zeta \varrho |\mathbf{u}|^2)^{\frac{1+\delta}{2}} |x - y|^{-1-\delta+\alpha} (\zeta \varrho |\mathbf{u}|^2)^{\frac{1-\delta}{2}} dy \quad \text{for } x \in B_R.$$

Applying the Hölder inequality we obtain

$$\mathcal{L}(x, t) \leq c \left( \int_{B_R} \zeta \varrho |\mathbf{u}|^2 |x - y|^{-2+\frac{2\alpha}{1+\delta}} dy \right)^{\frac{1+\delta}{2}} \left( \int_{B_R} \zeta \varrho |\mathbf{u}|^2 dy \right)^{\frac{1-\delta}{2}}.$$

This leads to the inequality

$$\begin{aligned} \int_{B_R} \mathcal{L}(x, t)^{\frac{2}{1+\delta}} dx &\leq \left( \int_{B_R} \zeta \varrho |\mathbf{u}|^2 dy \right)^{\frac{1-\delta}{1+\delta}} \int_{B_R} \int_{B_R} \zeta \varrho |\mathbf{u}|^2(y, t) |x - y|^{-2+\frac{2\alpha}{1+\delta}} dx dy \leq \\ &c \left( \int_{B_R} \zeta \varrho |\mathbf{u}|^2(y, t) dy \right)^{\frac{2}{1+\delta}} \leq c (\|\varrho |\mathbf{u}|^2(t)\|_{L^1(\Omega)})^{2/(1+\delta)}. \end{aligned}$$

Recalling the energy estimate (19) we finally obtain

$$\|\mathcal{L}\|_{L^\infty(0,T;L^{\frac{2}{1+\delta}}(B_R))} \leq c(R, \zeta) \|\varrho|\mathbf{u}|^2\|_{L^\infty(0,T;L^1(\Omega))} \leq c(\zeta, R)E. \quad (61)$$

Now our task is to estimate  $\mathcal{Q}$ . Notice that  $|x - y| \leq 2R$  for all  $x, y \in B_R$ . It follows from this and (12) that  $|x - y|^{-1} \leq c(R)G_1(x - y)$  for all  $x, y \in B_R$ , where  $G_1$  is the Bessel kernel. We thus get

$$\mathcal{Q}(x, t) \leq c(R) \int_{\mathbb{R}^2} G_1(x - y) \zeta \varrho(y, t) dy := G(x, t).$$

Estimate (13) for the Bessel kernel and inequality (55) yield

$$\|G\|_{L^2(0,T;H^{1/2}(\mathbb{R}^2))} \leq \|\zeta \varrho\|_{L(0,T;H^{-1/2}(\mathbb{R}^2))} \leq c(\zeta)E.$$

Since the embedding  $H^{1/2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$  is bounded, see [1], thm. 7.57, we obtain

$$\|\mathcal{Q}\|_{L^2(0,T;L^4(B_R))} \leq c(R) \|G\|_{L^2(0,T;L^4(B_R))} \leq c(R) \|G\|_{L^2(0,T;H^{1/2}(\mathbb{R}^2))} \leq c(R, \zeta)E.$$

Combining these inequalities with (61) we arrive at the estimates

$$\|\sqrt{\mathcal{L}}\|_{L^\infty(0,T;L^{\frac{4}{1+\delta}}(B_R))} \leq c(R, \zeta)E, \quad \|\sqrt{\mathcal{Q}}\|_{L^4(0,T;L^8(B_R))} \leq c(R, \zeta)E. \quad (62)$$

Next, the Hölder inequality implies that

$$\|\sqrt{\mathcal{L}}\sqrt{\mathcal{Q}}\|_{L^\tau(0,T;L^r(B_R))} \leq \|\sqrt{\mathcal{L}}\|_{L^{\tau_1}(0,T;L^{r_1}(B_R))} \|\sqrt{\mathcal{Q}}\|_{L^{\tau_2}(0,T;L^{r_2}(B_R))}$$

for all  $\tau, r, \tau_i, r_i \in [1, \infty]$  satisfying the condition

$$\tau^{-1} = \tau_1^{-1} + \tau_2^{-1}, \quad r^{-1} = r_1^{-1} + r_2^{-1}.$$

Setting  $\tau = \tau_2 = 4$ ,  $\tau_1 = \infty$ ,  $r_1 = 4/(1 + \delta)$ ,  $r_2 = 8$ ,  $r = 8/(3 + 2\delta)$ , and recalling inequalities (62) we obtain

$$\|\sqrt{\mathcal{L}}\sqrt{\mathcal{Q}}\|_{L^4(0,T;L^{8/(3+2\delta)}(B_R))} \leq c(\zeta, \delta, R)E.$$

Combining this result with (60) we arrive at (59).  $\square$

**Lemma 5.2.** . *Under the assumptions of Proposition 5.1, for every positive  $\nu < 3$  and  $R$ , there is a constant  $c(\nu, \zeta, R)$  such that*

$$\|\mathbf{F}\|_{L^2(0,T;L^{3-\nu}(B_R))} \leq c(\nu, \zeta, R)E. \quad (63)$$

*Proof.* Assume that  $\Omega \subset B_R$ . It follows from (12) that  $|x-y|^{-1} \leq c(R)G_1(x-y)$  for all  $x, y \in B_R$ . Thus we get

$$|\mathbf{F}(x, t)| \leq c \int_{\mathbb{R}^2} G_1(x-y) \zeta \varrho |\mathbf{u}|(y, t) dy := M(x, t) \quad \text{for all } x \in B_R. \quad (64)$$

Now choose an arbitrary  $\mu \in (0, 1/2)$ . It follows from (13) that

$$\|M(t)\|_{H^{1/2-\mu}(\mathbb{R}^2)} \leq \|\zeta \varrho |\mathbf{u}|(t)\|_{H^{-1/2-\mu}(\mathbb{R}^2)}.$$

Applying inequality (56) we obtain

$$\|M\|_{L^1(0,T;H^{1/2-\mu}(\mathbb{R}^2))} \leq c(\mu, \zeta)E.$$

Since the embedding  $H^{1/2-\mu}(\mathbb{R}^2) \hookrightarrow L^{\frac{4}{1+2\mu}}(\mathbb{R}^2)$  is bounded, [1], thm. 7.57, we get

$$\|M\|_{L^1(0,T;L^{\frac{4}{1+2\mu}}(\mathbb{R}^2))} \leq c(\mu, \zeta)E.$$

Combining this result with (64) we arrive at

$$\|\mathbf{F}\|_{L^1(0,T;L^{\frac{4}{1+2\mu}}(B_R))} \leq c(\mu, \zeta)E. \quad (65)$$

Next notice that by the interpolation inequality,

$$\|\mathbf{F}\|_{L^r(0,T;L^s(B_R))} \leq \|\mathbf{F}\|_{L^4(0,T;L^{\frac{8}{3+\delta}}(B_R))}^\alpha \|\mathbf{F}\|_{L^1(0,T;L^{\frac{4}{1+2\mu}}(B_R))}^{1-\alpha}$$

holds for all  $\alpha \in (0, 1)$  and

$$r^{-1} = \frac{\alpha}{4} + \frac{1-\alpha}{1}, \quad s^{-1} = \frac{3+\delta}{8}\alpha + \frac{1+2\mu}{4}(1-\alpha).$$

Setting  $\alpha = 2/3$  and recalling inequalities (59), (65) we obtain

$$\|\mathbf{F}\|_{L^2(0,T;L^{\frac{12}{4+\mu+\delta}}(B_R))} \leq c(\zeta, \delta, \mu, R).$$

Choosing  $\mu$  and  $\delta$  so small that  $3 - \nu < 12/(4 + \mu + \delta)$  we finally arrive at (63). □

## 6 $L^p$ estimates

In this section we investigate properties of solutions  $(\varrho, \mathbf{u})$  to regularized problem (18) and prove that the pressure function  $p(\varrho)$  is locally integrable with an exponent greater than 1. The corresponding result is given by the following

**Theorem 6.1.** *Let a solution  $(\varrho, \mathbf{u})$  to problem (18) meets all requirements of Proposition 3.1. Let  $\zeta \in C_0^\infty(\mathbb{R}^2)$  be an arbitrary nonnegative compactly supported in  $\Omega$  function and  $\lambda \in (0, 1/6)$ . Then*

$$\int_{Q_T} \zeta^2 p(\varrho) \varrho^\lambda dx dt \leq c(\zeta, \lambda) E, \quad (66)$$

where  $c(\zeta, \lambda)$  depends only on  $\zeta$  and  $\lambda$ .

The rest of the section is devoted to the proof of this theorem. Our strategy is the following. First we construct a special test function  $\boldsymbol{\xi}$  such that  $p \operatorname{div} \boldsymbol{\xi} \sim p(\varrho) \varrho^\lambda$ . Next we insert  $\boldsymbol{\xi}$  into (20) to obtain special integral identity containing the vector field  $\mathbf{F}$ . Finally we employ Lemmas 5.1 and 5.2 to obtain estimate (66). Hence the proof of Theorem 6.1 falls into four steps.

### 6.1 Step 1. Test functions

Fix an arbitrary  $\lambda \in (0, 1/6)$  and choose a function  $\psi \in C^\infty(\mathbb{R})$  with the properties

$$\psi(0) = 0, \quad \psi(s) \geq 0, \quad c^{-1}|s|^\lambda - 1 \leq \psi(s) \leq c|s|^\lambda, \quad |s\psi'(s)| \leq c|s|^\lambda, \quad (67)$$

where  $c$  is some positive constant. Next choose an arbitrary function  $\zeta \in C_0^\infty(\mathbb{R}^2)$  such that  $\zeta$  is nonnegative and is compactly supported in  $\Omega$ . Recall the definition of the mollifier  $[\cdot]_h$  and introduce the auxiliary function

$$g(x, t) = [\zeta \psi(\varrho)]_h(x, t) \quad \text{in } \mathbb{R}^2 \times [0, T]. \quad (68)$$

We will assume that  $h$  is less than the distance between the support of  $\zeta$  and the boundary of  $\Omega$ . Finally, introduce the test vector field

$$\boldsymbol{\xi}(x, t) = \zeta(x) \mathbf{H}(x, t), \quad \text{where } \mathbf{H} = \nabla \Delta^{-1} g. \quad (69)$$

The following lemmas constitute the basic properties of  $\psi(\varrho)$ ,  $g$ , and  $\mathbf{H}$ .

**Lemma 6.1.** *Under the assumptions of Theorem 6.1, there is a constant  $c(\lambda)$ , depending only on  $\lambda$  and  $\psi$ , such that*

$$\|\psi(\varrho)\nabla\mathbf{u}\|_{L^2(0,T;L^{\frac{2}{1+2\lambda}}(\mathbb{R}^2))} + \|(\psi(\varrho) - \varrho\psi'(\varrho))\nabla\mathbf{u}\|_{L^2(0,T;L^{\frac{2}{1+2\lambda}}(\mathbb{R}^2))} \leq c(\lambda)E, \quad (70)$$

$$\|\psi(\varrho)\mathbf{u}\|_{L^2(0,T;L^3(\mathbb{R}^2))} \leq c(\lambda)E. \quad (71)$$

*Proof.* Notice that

$$|\psi(\varrho)\nabla\mathbf{u}| + |(\psi(\varrho) - \varrho\psi'(\varrho))\nabla\mathbf{u}| \leq c\varrho^\lambda|\nabla\mathbf{u}|. \quad (72)$$

Recall that  $\mathbf{u}$  and  $\varrho$  vanish outside of  $\Omega \times [0, T]$ . From this and relations

$$1/2 + 1/\infty = 1/2, \quad 1/2 + 1/(1/\lambda) = (1 + 2\lambda)/2$$

we obtain

$$\begin{aligned} \|\varrho^\lambda\nabla\mathbf{u}\|_{L^2(0,T;L^{\frac{2}{1+2\lambda}}(\Omega))} &\leq \|\nabla\mathbf{u}\|_{L^2(0,T;L^2(\Omega))} \|\varrho^\lambda\|_{L^\infty(0,T;L^{1/\lambda}(\Omega))} \leq \\ &c(\lambda)\|\nabla\mathbf{u}\|_{L^2(0,T;L^2(\Omega))} \|\varrho\|_{L^\infty(0,T;L^1(\Omega))}^\lambda \leq c(\lambda)E, \end{aligned}$$

which along with (72) yields (70). Next set  $q = 3/(1 - 3\lambda)$ . Since  $1/q + 1/(1/\lambda) = 1/3$ , we have

$$\begin{aligned} \|\psi(\varrho)\mathbf{u}\|_{L^2(0,T;L^3(\mathbb{R}^2))} &\leq c\|\varrho^\lambda\mathbf{u}\|_{L^2(0,T;L^3(\Omega))} \leq c\|\varrho^\lambda\|_{L^\infty(0,T;L^{1/\lambda}(\Omega))}\|\mathbf{u}\|_{L^2(0,T;L^q(\Omega))} \\ &\leq c(\lambda)\|\varrho\|_{L^\infty(0,T;L^1(\Omega))}^\lambda\|\mathbf{u}\|_{L^2(0,T;L^q(\Omega))} \leq c(\lambda)\|\varrho\|_{L^\infty(0,T;L^1(\Omega))}^\lambda\|\mathbf{u}\|_{L^2(0,T;W_0^{1,2}(\Omega))} \leq cE, \end{aligned}$$

and the lemma follows.  $\square$

**Lemma 6.2.** *Under the assumptions of Theorem 6.1, the function  $g$  belongs to the class  $L^\infty(0, T; C^k(\mathbb{R}^2))$  for every integer  $k \geq 0$ . It is compactly supported in  $\Omega \times [0, T]$  and admits the estimate*

$$\|g\|_{L^\infty(0,T;L^{1/\lambda}(\mathbb{R}^2))} \leq c(\zeta)E. \quad (73)$$

Moreover,  $\partial_t g$  belongs to the class  $L^2(0, T; C^k(\mathbb{R}^2))$  and has the representation

$$\partial_t g = -\operatorname{div} [\zeta\psi(\varrho)\mathbf{u}]_h + [\psi(\varrho)\nabla\zeta\mathbf{u}]_h + [\zeta(\psi(\varrho) - \psi'(\varrho)\varrho) \operatorname{div} \mathbf{u}]_h. \quad (74)$$

*Proof.* Since  $\zeta\psi(\varrho) \in L^\infty(0, T; L^1(\mathbb{R}^2))$ , it follows from general properties of the mollifier that  $g \in L^\infty(0, T; C^k(\mathbb{R}^2))$ . Since  $h$  is less than the distance between  $\text{spt } \zeta$  and  $\mathbb{R}^2 \setminus \Omega$ , the function  $g$  is supported in  $\Omega \times [0, T]$ . Next, inequality (67) implies the estimate

$$\|g(t)\|_{L^{1/\lambda}(\mathbb{R}^2)} \leq \|\zeta\psi(\varrho)(t)\|_{L^{1/\lambda}(\mathbb{R}^2)} \leq c(\zeta)\|\varrho(t)\|_{L^1(\mathbb{R}^2)}^\lambda,$$

which along with energy inequality (19) yields (73). Let us consider the time derivative of  $g$ . In view of (22) the integral identity

$$\int_{Q_T} \left( \psi(\varrho) \partial_t \varsigma + (\psi(\varrho) \mathbf{u}) \cdot \nabla \varsigma - \varsigma (\psi'(\varrho) \varrho - \psi(\varrho)) \operatorname{div} \mathbf{u} \right) dx dt = 0 \quad (75)$$

holds for every function  $\varsigma \in C^\infty(\Pi)$  which is supported in  $Q_T$ . Choose an arbitrary  $\xi \in C_0^\infty(0, T)$ ,  $y \in \mathbb{R}^2$ , and set

$$\varsigma(x, t) = \xi(t) \zeta(x) h^{-2} \omega\left(\frac{x_1 - y_1}{h}\right) \omega\left(\frac{x_2 - y_2}{h}\right).$$

Substituting  $\varsigma$  into (75) we obtain

$$\begin{aligned} \int_0^T \xi'(t) g(y, t) dt - \int_0^T \xi(t) \operatorname{div} [\zeta \psi(\varrho) \mathbf{u}]_h(y, t) dt + \int_0^T \xi(t) [\psi(\varrho) \nabla \zeta \mathbf{u}]_h(y, t) dt + \\ \int_0^T \xi(t) [\zeta (\psi(\varrho) - \psi'(\varrho) \varrho) \operatorname{div} \mathbf{u}]_h(y, t) dt = 0, \end{aligned}$$

which yields (74). In view of Lemma 6.1, the functions  $\psi(\varrho) \zeta \mathbf{u}$ ,  $\psi(\varrho) \nabla \zeta \cdot \mathbf{u}$  belong to the class  $L^2(0, T; L^3(\mathbb{R}^2))$ , and the function  $\zeta (\psi(\varrho) - \psi'(\varrho) \varrho) \operatorname{div} \mathbf{u}$  belong to  $L^2(0, T; L^{2/(1+2\lambda)}(\mathbb{R}^2))$ . Since the mollifier  $[\cdot]_h : L^p(\mathbb{R}^2) \rightarrow C^k(\mathbb{R}^2)$  is bounded for all  $p \geq 1$  and  $k \geq 0$ , the representation (35) yields the inclusion  $\partial_t g \in L^2(0, T; C^k(\mathbb{R}^2))$ .  $\square$

**Lemma 6.3.** *Under the assumptions of Theorem 6.1,  $\mathbf{H}$  belongs to the class  $L^\infty(0, T; C^k(\mathbb{R}^2))$ , and  $\partial_t \mathbf{H}$  belongs to the class  $L^2(0, T; C^k(\mathbb{R}^2))$  for every integer  $k \geq 0$ . Moreover,  $\mathbf{H}$  admits the estimates.*

$$\|\mathbf{H}\|_{L^\infty(0, T; L^\infty(\Omega))} \leq c(\zeta) E, \quad \|\nabla \mathbf{H}\|_{L^\infty(0, T; L^{1/\lambda}(\Omega))} \leq c(\zeta) E. \quad (76)$$

*Proof.* Since the function  $g$  is supported in  $Q_T$ , the inclusions  $\mathbf{H} \in L^\infty(0, T; C^k(\mathbb{R}^2))$  and  $\partial_t \mathbf{H} \in L^2(0, T; C^k(\mathbb{R}^2))$  obviously follow from (6.2). Now choose an arbitrary  $R$  such that  $\Omega \Subset B_{R/2}$ . Since  $\text{spt } g(t) \subset \Omega$ , we have

$$\|\mathbf{H}(t)\|_{W^{1/\lambda}(B_R)} \leq c(R, \Omega) \|g(t)\|_{L^{1/\lambda}(\Omega)}$$



The embedding  $W^{1/\lambda}(B_R) \hookrightarrow C(B_R)$  is bounded for every  $\lambda < 1/2$ . It follows that

$$\begin{aligned} \|\mathbf{H}\|_{L^\infty(0,T;C(B_R))} &\leq c(R, \lambda) \|\mathbf{H}\|_{L^\infty(0,T;W^{1/\lambda}(B_R))} \leq \\ &\|g\|_{L^\infty(0,T;L^{1/\lambda}(\Omega))} \leq c(R, \zeta, \lambda) E \end{aligned} \quad (77)$$

Since  $\Omega \subset B_{R/2}$ , representation (17) implies

$$|\mathbf{H}(x, t)| \leq c(R, \Omega) \|g(t)\|_{L^1(\Omega)} \quad \text{for } x \in \mathbb{R}^2 \setminus B_R$$

Thus we get

$$\|\mathbf{H}\|_{L^\infty(0,T;C(\mathbb{R}^2 \setminus B_R))} \leq c(R, \lambda) \|g\|_{L^\infty(0,T;L^1(\Omega))} \leq c(R, \zeta, \lambda) E.$$

Combining this result with (77) gives the first inequality in (76). Next notice that

$$\partial_{x_i} H_j = \partial_{x_i} \partial_{x_j} \Delta^{-1} g = R_i R_j g,$$

where  $R_i, R_j$  are the Riesz singular operators. Since the Riesz operators are bounded in  $L^{1/\lambda}(\mathbb{R}^2)$ , the second inequality in (76) is a straightforward consequence of estimate (73). □

## 6.2 Step 2. Integral identities

The proof of Theorem 6.1 is based on the special integral identity which is given by the following proposition.

**Proposition 6.1.** *Under the assumptions of Theorem 6.1, we have*

$$\int_{\Pi} \zeta p(\varrho) [\zeta \psi(\varrho)]_h dx dt = \sum_{i=1}^7 \Gamma^{(i)} + I_h, \quad (78)$$

where

$$\begin{aligned}
\Gamma^{(1)} &= \int_{Q_T} F_1 [\zeta(\psi(\varrho) - \varrho\psi'(\varrho)) \operatorname{div} \mathbf{u}]_h dxdt, \\
\Gamma^{(3)} &= \int_{Q_T} \left( gF_1 \operatorname{div} \mathbf{u} - F_1 \nabla u_j \cdot \frac{\partial \mathbf{H}}{\partial x_j} - F_2 \nabla^\perp u_j \cdot \frac{\partial \mathbf{H}}{\partial x_j} \right) dxdt, \\
\Gamma^{(2)} &= \int_{Q_T} F_1 [\psi(\varrho) \nabla \zeta \cdot \mathbf{u}]_h dxdt, \quad \Gamma^{(4)} = - \int_{Q_T} \varrho(\mathbf{u} \cdot \nabla \zeta) (\mathbf{u} \cdot \mathbf{H}) dxdt, \\
\Gamma^{(5)} &= - \int_{Q_T} \zeta p \nabla \zeta \cdot \mathbf{H} dxdt, \quad \Gamma^{(6)} = \int_{Q_T} (\mathbb{S}(\mathbf{u}) : \nabla \boldsymbol{\xi} - \varrho \mathbf{f} \cdot \boldsymbol{\xi}) dxdt, \\
\Gamma^{(7)} &= \lim_{\tau \rightarrow 0} \frac{1}{\tau} \int_{T-\tau}^T \int_{\Omega} \zeta \varrho \mathbf{u} \cdot \mathbf{H} dxdt - \int_{\Omega} \varrho_0(x) \zeta \mathbf{u}_0(x) \mathbf{H}(x, 0) dx
\end{aligned} \tag{79}$$

$$I_h = - \int_{\mathbb{R}^2 \times [0, T]} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1} ([\zeta \psi(\varrho)]_h \mathbf{u} - [\zeta \psi(\varrho) \mathbf{u}]_h) dxdt. \tag{80}$$

Here  $\mathbf{F}$  is a solution to the Cauchy-Riemann equations (57), and  $\mathbf{H}$  is given by (69).

*Proof.* Recall formulae (32) and (69) for the cut-off function  $\eta_\tau$  and the vector field  $\boldsymbol{\xi}$ . Notice that the function  $\eta_\tau \boldsymbol{\xi}$  and its time derivative belong to  $L^\infty(0, T; C^k(\Omega))$  and  $L^2(0, T; C^k(\Omega))$  respectively for all integer  $k \geq 0$ . Moreover,  $\eta_\tau \boldsymbol{\xi}$  vanishes at the lateral side and the top of the cylinder  $Q_T$ . Hence we can use this function as a test function in integral identity (20) to obtain

$$\begin{aligned}
&\int_{Q_T} \eta_\tau(t) (\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi}) dxdt + \int_{Q_T} \eta_\tau (\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} - \mathbb{S}(\mathbf{u})) : \nabla \boldsymbol{\xi} dxdt + \\
&\quad \int_{Q_T} \eta_\tau \varrho \mathbf{f} \cdot \boldsymbol{\xi} dxdt = \Gamma_T(\tau), \tag{81}
\end{aligned}$$

where

$$\Gamma_T(\tau) = \frac{1}{\tau} \int_{T-\tau}^T \int_{\Omega} \zeta \varrho \mathbf{u} \cdot \mathbf{H} dxdt - \int_{\Omega} \varrho_0(x) \zeta \mathbf{u}_0(x) \mathbf{H}(x, 0) dx \tag{82}$$

Letting  $\tau \rightarrow 0$  in (81) we arrive at

$$\int_{Q_T} (\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi}) dxdt + \int_{Q_T} (\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} - \mathbb{S}(\mathbf{u})) : \nabla \boldsymbol{\xi} dxdt + \int_{Q_T} \varrho \mathbf{f} \cdot \boldsymbol{\xi} dxdt = \Gamma^{(7)}.$$

The limit  $\Gamma^{(7)} = \lim_{\tau \rightarrow 0} \Gamma_T(\tau)$  exists since there exists the limit of the left hand side of (81). We can rewrite the latter identity in the form

$$\int_{Q_T} p(\varrho) \operatorname{div} \boldsymbol{\xi} \, dxdt = \Gamma^{(6)} + \Gamma^{(7)} - \int_{Q_T} \varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi} \, dxdt - \int_{Q_T} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\xi} \, dxdt. \quad (83)$$

It follows from the expression (69) for  $\boldsymbol{\xi}$  that  $\partial_t \boldsymbol{\xi} = \zeta \nabla \Delta^{-1} \partial_t g$ , where  $g$  is given by (68). From this and the representation (74) in Lemma 6.2 we obtain the identity

$$\begin{aligned} \int_{Q_T} \varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi} \, dxdt &= \int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} [\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \operatorname{div} \mathbf{u}]_h \, dxdt + \\ &\int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} [\psi(\varrho) \nabla \zeta \cdot \mathbf{u}]_h \, dxdt - \int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1} [\zeta \psi(\varrho) \mathbf{u}]_h \, dxdt \end{aligned} \quad (84)$$

Since  $\zeta$ ,  $[\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \operatorname{div} \mathbf{u}]_h$ , and  $[\psi(\varrho) \nabla \zeta \cdot \mathbf{u}]_h$  are supported in  $Q_T$ , we have

$$\begin{aligned} \int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} [\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \operatorname{div} \mathbf{u}]_h \, dxdt &= \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} [\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \operatorname{div} \mathbf{u}]_h \, dxdt \\ &= - \int_{\Pi} F_1 [\zeta(\psi(\varrho) - \varrho \psi'(\varrho)) \operatorname{div} \mathbf{u}]_h \, dxdt = -\Gamma^{(1)}, \\ \int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} [\psi(\varrho) \nabla \zeta \cdot \mathbf{u}]_h \, dxdt &= \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \Delta^{-1} [\psi(\varrho) \nabla \zeta \cdot \mathbf{u}]_h \, dxdt = \\ &= - \int_{\Pi} F_1 [\psi(\varrho) \nabla \zeta \cdot \mathbf{u}]_h \, dxdt = -\Gamma^{(2)}. \end{aligned}$$

Inserting these equalities into (84) we arrive at

$$\int_{\Pi} \varrho \mathbf{u} \cdot \partial_t \boldsymbol{\xi} \, dxdt = -\Gamma^{(1)} - \Gamma^{(2)} - \int_{Q_T} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1} [\zeta \psi(\varrho) \mathbf{u}]_h \, dxdt. \quad (85)$$

Next, expression (69) for  $\mathbf{H}$  implies

$$\begin{aligned} \int_{Q_T} \varrho u_i u_j \frac{\partial \xi_i}{\partial x_j} \, dxdt &= \int_{\Pi} u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot (\zeta \varrho \mathbf{u}) \, dxdt + \\ &\int_{\Pi} \varrho (\mathbf{u} \cdot \nabla \zeta) (\mathbf{u} \cdot \mathbf{H}) \, dxdt = \int_{\Pi} u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot (\zeta \varrho \mathbf{u}) \, dxdt - \Gamma^{(4)} \end{aligned} \quad (86)$$

Using equation (57) we obtain the identity

$$\int_{\Pi} u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot (\zeta \varrho \mathbf{u}) \, dx dt = \int_{\Pi} u_j (\nabla F_1 + \nabla^\perp F_2) \cdot \frac{\partial \mathbf{H}}{\partial x_j} \, dx dt.$$

In view of Corollary 3.1, the function  $\varrho \mathbf{u}$  belongs to the class  $L^2(\Pi)$ . Obviously it is supported in  $\Omega \times (0, T)$ . From this and formula (59) we conclude that  $\mathbf{F} \in L^2(0, T; W^{1,2}(B_R))$  for every ball  $B_R \subset \mathbb{R}^2$ . Recall that  $\mathbf{u} \in L^2(0, T; W^{1,2}(\mathbb{R}^2))$  is compactly supported in  $\Omega \times (0, T)$ . Finally notice that  $\mathbf{H} \in L^\infty(0, T; C^k(\mathbb{R}^2))$  for every  $k \geq 0$ . Hence we can integrate by parts to obtain

$$\begin{aligned} \int_{\Pi} u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot (\zeta \varrho \mathbf{u}) \, dx dt &= \int_{\Pi} \left( F_2 \operatorname{rot} \frac{\partial \mathbf{H}}{\partial x_j} - F_1 \operatorname{div} \frac{\partial \mathbf{H}}{\partial x_j} \right) u_j \, dx dt - \\ &\quad \int_{\Pi} \left( F_1 \nabla u_j \cdot \frac{\partial \mathbf{H}}{\partial x_j} + F_2 \nabla^\perp u_j \cdot \frac{\partial \mathbf{H}}{\partial x_j} \right) \, dx dt. \end{aligned} \quad (87)$$

Recall that  $\mathbf{H} = \nabla \Delta^{-1} g$ , where  $g = [\zeta \psi(\varrho)]_h \in L^\infty(0, T; C^3(\mathbb{R}^2))$  is supported in  $Q_T$ . It follows that

$$\operatorname{rot} \frac{\partial \mathbf{H}}{\partial x_j} = 0, \quad \operatorname{div} \frac{\partial \mathbf{H}}{\partial x_j} = \partial_{x_j} [\zeta \psi(\varrho)]_h.$$

We thus get

$$\begin{aligned} \int_{\Pi} \left( F_2 \operatorname{rot} \frac{\partial \mathbf{H}}{\partial x_j} - F_1 \operatorname{div} \frac{\partial \mathbf{H}}{\partial x_j} \right) u_j \, dx dt &= - \int_{\Pi} F_1 u_j \partial_{x_j} [\zeta \psi(\varrho)]_h = \\ &= - \int_{Q_T} F_1 \operatorname{div} ([\zeta \psi(\varrho)]_h \mathbf{u}) \, dx dt + \int_{Q_T} g F_1 \operatorname{div} \mathbf{u} \, dx dt. \end{aligned}$$

Noting that  $F_1 = \operatorname{div} \Delta^{-1}(\zeta \varrho \mathbf{u})$  we arrive at the identity

$$\begin{aligned} \int_{\Pi} \left( F_2 \operatorname{rot} \frac{\partial \mathbf{H}}{\partial x_j} - F_1 \operatorname{div} \frac{\partial \mathbf{H}}{\partial x_j} \right) u_j \, dx dt &= \\ &= \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1}([\zeta \psi(\varrho)]_h \mathbf{u}) \, dx dt + \int_{\Pi} g F_1 \operatorname{div} \mathbf{u} \, dx dt. \end{aligned}$$

Inserting this equality into (87) and recalling the expression (81) for  $\Gamma^{(3)}$  we get

$$\int_{\Pi} \zeta \varrho u_j \frac{\partial \mathbf{H}}{\partial x_j} \cdot \mathbf{u} \, dx dt = -\Gamma^{(3)} - \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1}([\zeta \psi(\varrho)]_h \mathbf{u}) \, dx dt.$$

Inserting this result into (86) we finally obtain

$$\int_{\Pi} \varrho u_i u_j \frac{\partial \xi_i}{\partial x_j} dx dt = -\Gamma^{(3)} - \Gamma^{(4)} + \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1}([\zeta \psi(\varrho)]_h \mathbf{u}) dx dt. \quad (88)$$

It remains to note that in view of (79) and (69) we have

$$\int_{\Pi} p \operatorname{div} \boldsymbol{\xi} dx dt = \int_{\Pi} \zeta p [\zeta \psi(\varrho)]_h dx dt - \Gamma^{(5)}. \quad (89)$$

Inserting (85), (88), and (89) into (83) we obtain the desired identity (78).  $\square$

### 6.3 Step 3. Estimates of $\Gamma^{(i)}$

In this section we show that the quantities  $\Gamma^{(i)}$  in the basic integral identity (78) are bounded by a constant, depending only on the exponent  $\lambda$ , the cut-off function  $\zeta$ , and the constant  $E$  specified by Remark 1.1.

**Proposition 6.2.** *Under the assumptions of Theorem 6.1,*

$$\Gamma^{(i)} \leq c(\zeta, \lambda) E, \quad (90)$$

where  $c$  depends only on  $\zeta$  and  $\lambda$ .

*Proof.* Let us estimate  $\Gamma^{(1)}$  and  $\Gamma^{(2)}$ . Since  $\lambda < 1/6$ , we can choose  $\nu > 0$ , depending on  $\lambda$ , such that  $(3 - \nu)^{-1} + (1 + 2\lambda)2^{-1} = 1$ . Lemmas 5.2, 6.1 and the Hölder inequality imply

$$\begin{aligned} |\Gamma^{(1)}| &\leq \|F_1\|_{L^2(0,T;L^{3-\nu}(\mathbb{R}^2))} \|\zeta(\psi(\varrho) - \varrho\psi'(\varrho)) \operatorname{div} \mathbf{u}\|_{L^2(0,T;L^{2/(1+2\lambda)}(\mathbb{R}^2))} \leq \\ &\|F_1\|_{L^2(0,T;L^{3-\nu}(\mathbb{R}^2))} \|\zeta(\psi(\varrho) - \varrho\psi'(\varrho)) \operatorname{div} \mathbf{u}\|_{L^2(0,T;L^{2/(1+2\lambda)}(\mathbb{R}^2))} \leq c(\lambda, \zeta) E. \end{aligned} \quad (91)$$

Applying Lemmas 5.2 and 6.1 once more we obtain

$$\begin{aligned} |\Gamma^{(2)}| &\leq \|F_1\|_{L^2(0,T;L^{3-\nu}(\mathbb{R}^2))} \|\psi(\varrho) \nabla \zeta \cdot \mathbf{u}\|_{L^2(0,T;L^2(\mathbb{R}^2))} \leq \\ &\|F_1\|_{L^2(0,T;L^{3-\nu}(\mathbb{R}^2))} \|\psi(\varrho) \nabla \zeta \cdot \mathbf{u}\|_{L^2(0,T;L^2(\mathbb{R}^2))} \leq c(\lambda, \zeta) E. \end{aligned} \quad (92)$$

Our next task is to estimate  $\Gamma^{(3)}$  and  $\Gamma^{(4)}$ . Since  $\mathbf{u}$  is supported in  $\Omega \times [0, T]$ , we have

$$|\Gamma^{(3)}| \leq \int_{\Omega \times (0,T)} |\mathbf{F}| |\nabla \mathbf{u}| (|g| + |\nabla \mathbf{H}|) dx dt \quad (93)$$

It follows from Lemmas 6.2 and 6.3 that

$$\|g\|_{L^\infty(0,T;L^{1/\lambda}(\mathbb{R}^2))} + \|\nabla \mathbf{H}\|_{L^\infty(0,T;L^{1/\lambda}(\mathbb{R}^2))} \leq c(\zeta, \lambda)E. \quad (94)$$

On the other hand, energy estimate (19) and Lemma 5.2 imply

$$\|\mathbf{F}\|_{L^2(0,T;L^{3-\kappa}(\Omega))} \leq c(\zeta, \kappa)E, \quad \|\nabla \mathbf{u}\|_{L^2(0,T;L^2(\Omega))} \leq c(\zeta, \lambda)E, \quad (95)$$

where  $\kappa$  is an arbitrary positive number. Since  $\lambda < 1/6$ , we can choose  $\kappa$  such that  $(3 - \kappa)^{-1} + \lambda + 2^{-1} = 1$ . Applying the Hölder inequality and using (94), (95) we obtain

$$\begin{aligned} & \int_{\Omega \times (0,T)} |\mathbf{F}| |\nabla \mathbf{u}| (|g| + |\nabla \mathbf{H}|) dxdt \leq \\ & \|\mathbf{F}\|_{L^2(0,T;L^{3-\kappa}(\Omega))} \|\nabla \mathbf{u}\|_{L^2(0,T;L^2(\Omega))} (\|g\|_{L^\infty(0,T;L^{1/\lambda}(\mathbb{R}^2))} + \|\nabla \mathbf{H}\|_{L^\infty(0,T;L^{1/\lambda}(\mathbb{R}^2))}) \leq c(\zeta, \lambda)E. \end{aligned}$$

which leads to estimate (90) for  $\Gamma^{(3)}$ . In order to estimate  $\Gamma^{(4)}$  notice that in view of Lemma 6.3 and energy estimate (19), we have

$$\begin{aligned} |\Gamma^{(4)}| & \leq c(\zeta) \int_{\Omega \times (0,T)} |\mathbf{H}| \varrho |\mathbf{u}|^2 dxdt \leq \\ & \leq c(\zeta) \|\mathbf{H}\|_{L^\infty(\Omega \times (0,T))} \int_{\Omega \times (0,T)} \varrho |\mathbf{u}|^2 dxdt \leq c(\zeta, \lambda)E. \end{aligned}$$

Next we employ Lemma 6.3 and estimate (19) to obtain

$$|\Gamma^{(5)}| \leq \int_{\mathbb{R}^2 \times (0,T)} \zeta p |\nabla \zeta| |\mathbf{H}| dxdt \leq c(\zeta) \|\mathbf{H}\|_{L^\infty(\Omega \times (0,T))} \int_{\Omega \times (0,T)} p dxdt \leq c(\zeta, \lambda)E.$$

It remains to estimate  $\Gamma^{(6)}$  and  $\Gamma^{(7)}$ . Expression (69) for the vector field  $\boldsymbol{\xi}$ , and expression (79) for  $\Gamma^{(6)}$  yield the estimate

$$\begin{aligned} |\Gamma^{(6)}| & \leq \left| \int_{\Pi} \mathbb{S}(\mathbf{u}) : \nabla \boldsymbol{\xi} dxdt \right| + \left| \int_{\Pi} \varrho \mathbf{f} \cdot \boldsymbol{\xi} dxdt \right| \leq \\ & \int_{\Pi} \zeta |\mathbb{S}(\mathbf{u})| |H| dxdt + \int_{\Pi} (\zeta + |\nabla \zeta|) |\mathbf{H}| (|\mathbb{S}(\mathbf{u})| + \varrho |\mathbf{f}|) dxdt \leq \\ & c(\zeta) \int_{Q_T} |\nabla \mathbf{u}| (|\mathbf{H}| + |\nabla \mathbf{H}|) dxdt + c(\zeta)E \int_{Q_T} \varrho |\mathbf{H}| dxdt. \quad (96) \end{aligned}$$

On the other hand, energy estimate (19) yields

$$\int_{Q_T} |\nabla \mathbf{u}|(|\mathbf{H}| + |\nabla \mathbf{H}|) dxdt \leq c(\zeta) \|\nabla \mathbf{u}\|_{L^2(Q_T)} \|\mathbf{H}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(\zeta) E \|\mathbf{H}\|_{L^2(0,T;W^{1,2}(\Omega))}$$

Next, Lemma 6.3 and the inequality  $1/\lambda > 2$  imply

$$\|\mathbf{H}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq \|\mathbf{H}\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\nabla \mathbf{H}\|_{L^\infty(0,T;L^{1/\lambda}(\Omega))} \leq c(\zeta, \lambda) E.$$

We thus get

$$\int_{Q_T} |\nabla \mathbf{u}|(|\mathbf{H}| + |\nabla \mathbf{H}|) dxdt \leq c(\zeta, \lambda) E \quad (97)$$

Finally notice that

$$\int_{Q_T} \varrho |\mathbf{H}| dxdt \leq \|\mathbf{H}\|_{L^\infty(Q_T)} \|\varrho\|_{L^\infty(0,T;L^1(\Omega))} \leq c(\zeta, \lambda) E.$$

Inserting this estimate along with (97) into (96) we arrive at the desired estimate (90) for  $\Gamma^{(6)}$ . Finally, expression (82) for  $\Gamma_T$ , Lemma 6.3, and the energy estimate (19) imply

$$|\Gamma_T(\tau)| \leq c(\zeta) \|\mathbf{H}\|_{L^\infty(Q_T)} \|\varrho\|_{L^\infty(0,T;L^1(\Omega))} \leq c(\zeta) E. \quad (98)$$

It follows from this that  $\Gamma^{(7)} = \lim_{\tau \rightarrow 0} \Gamma_T(\tau)$  satisfies inequality (90).  $\square$

## 6.4 Step 4. Proof of Theorem 6.1

The proof is based on Propositions 6.1 and 6.2. First we show that the quantity  $I_h$  in identity (78) tends to zero as  $h \rightarrow 0$ . We begin with the observation that

$$\begin{aligned} I_h &= \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1} \left( [\zeta \psi(\varrho) \mathbf{u}]_h - \zeta \psi(\varrho) \mathbf{u} \right) dxdt - \\ &\quad \int_{\Pi} \zeta \varrho \mathbf{u} \cdot \nabla \operatorname{div} \Delta^{-1} \left( ([\zeta \psi(\varrho)]_h - \zeta \psi(\varrho)) \mathbf{u} \right) dxdt. \end{aligned}$$

Since the Riesz operator  $\nabla \operatorname{div} \Delta^{-1} : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$  is bounded and  $\zeta$  is supported in  $\Omega$ , we have

$$|I_h|^2 \leq c \|\varrho \mathbf{u}\|_{L^2(Q_T)}^2 \int_0^T (J_h(t) + L_h(t)) dt, \quad (99)$$

where

$$J_h(t) = \| [\zeta\psi(\varrho)\mathbf{u}]_h(t) - \zeta\psi(\varrho)\mathbf{u}(t) \|_{L^2(\Omega)}^2, \quad L_h(t) = \| [\zeta\psi(\varrho)]_h\mathbf{u}(t) - \zeta\psi(\varrho)\mathbf{u}(t) \|_{L^2(\Omega)}^2.$$

In view of Corollary 3.2 the vector field  $\varrho\mathbf{u}$  belongs to the class  $L^2(Q_T)$ . Hence it suffices to prove that the sequence  $J_h + L_h$  has an integrable majorant and tends to 0 a.e. in  $(0, T)$  as  $h \rightarrow 0$ . Inequality (71) in Lemma 6.1 implies that  $\zeta\psi(\varrho)\mathbf{u} \in L^2(0, T; L^2(\mathbb{R}^2))$ . It follows from properties of the mollifier that  $J_h(t) \rightarrow 0$  as  $h \rightarrow 0$  a.e. in  $(0, T)$ . On the other hand, we have

$$J_h(t) \leq \| [\psi(\varrho)\mathbf{u}]_h(t) \|_{L^2(\Omega)} + \| \psi(\varrho)\mathbf{u}(t) \|_{L^2(\Omega)} \leq 2\| \psi(\varrho)\mathbf{u}(t) \|_{L^2(\Omega)}$$

By (71), the right hand side is integrable over  $[0, T]$  and is independent of  $h$ . Hence the sequence  $J_h \rightarrow 0$  a.e. in  $(0, T)$  and has an integrable majorant. Next, the Hölder inequality implies

$$L_h(t) \leq \| [\zeta\psi(\varrho)]_h(t) - \zeta\psi(\varrho)(t) \|_{L^{1/\lambda}(\Omega)}^2 \| \mathbf{u}(t) \|_{L^{2/(1-2\lambda)}(\Omega)}^2$$

Since the embedding  $W_0^{1,2}(\Omega) \hookrightarrow L^{2/(1-2\lambda)}(\Omega)$  is bounded, we have

$$L_h(t) \leq c(\lambda) \| [\zeta\psi(\varrho)]_h(t) - \zeta\psi(\varrho)(t) \|_{L^{1/\lambda}(\Omega)}^2 \| \mathbf{u}(t) \|_{W_0^{1,2}(\Omega)}^2. \quad (100)$$

Since  $\zeta\psi(\varrho) \in L^\infty(0, T; L^{1/\lambda}(\Omega))$ , we have

$$\| [\zeta\psi(\varrho)]_h(t) - \zeta\psi(\varrho)(t) \|_{L^{1/\lambda}(\Omega)}^2 \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad \text{for a.e. } t \in (0, T).$$

Hence  $L_h(t) \rightarrow 0$  a.e. in  $(0, T)$ . Notice that

$$\begin{aligned} \| [\zeta\psi(\varrho)]_h(t) - \zeta\psi(\varrho)(t) \|_{L^{1/\lambda}(\Omega)} &\leq \\ &\| [\zeta\psi(\varrho)]_h(t) \|_{L^{1/\lambda}(\Omega)} + \| \zeta\psi(\varrho)(t) \|_{L^{1/\lambda}(\Omega)} \leq 2\| \zeta\psi(\varrho)(t) \|_{L^{1/\lambda}(\Omega)}. \end{aligned}$$

Combing this result with (100) and recalling that  $\zeta\psi(\varrho) \in L^\infty(0, T; L^{1/\lambda}(\Omega))$  we obtain  $L_h(t) \leq c\| \mathbf{u}(t) \|_{W_0^{1,2}(\Omega)}^2$ . In view of the energy estimate (19) the right side of this inequality is integrable over  $(0, T)$ . Hence the sequence  $L_h$  has an integrable majorant. Applying the Lebesgue dominant convergence Theorem we arrive at the relation

$$\int_0^T (J_h(t) + L_h(t)) dt \rightarrow 0 \quad \text{as } h \rightarrow 0.$$



From this and (99) we conclude that  $I_h \rightarrow 0$  as  $h \rightarrow 0$ . Next, Propositions 6.1 and 6.2 imply

$$\int_{Q_T} \zeta p[\zeta \psi(\varrho)]_h dxdt \leq I_h + c(\zeta, \lambda)E. \quad (101)$$

The functions  $[\zeta \psi(\varrho)]_h$  are nonnegative and converge a.e. in  $Q_T$  to  $\zeta \psi(\varrho)$ . Letting  $h \rightarrow 0$  in (101) and applying the Fatou Theorem we arrive at the inequality

$$\int_{Q_T} \zeta^2 p\psi(\varrho) dxdt \leq c(\zeta, \lambda)E.$$

It remains to note that  $\psi(\varrho) \geq c\varrho^\lambda - 1$  and the theorem follows.

## 7 Proof of Theorems 1.1 and 1.2

### 7.1 Proof of Theorem 1.1

By Proposition 3.1, for every  $\varepsilon > 0$  regularized problem (18) has a solution  $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$  which admits estimates (19) and satisfies integral identities (20), (21).

**Lemma 7.1.** *Let  $\lambda \in [0, 1/6)$  and  $\Omega' \Subset \Omega$ . Then there are exponent  $r, p \in (2, \infty)$  and  $q, s, z \in (1, \infty)$  such that*

$$\|\varrho_\varepsilon\|_{L^{1+\lambda}(\Omega' \times (0, T))} + \varepsilon \int_{\Omega' \times (0, T)} \varrho_\varepsilon^{\gamma+\lambda} \leq C, \quad (102)$$

$$\|\varrho_\varepsilon\|_{L^r(0, T; L^s(\Omega'))} + \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^p(0, T; L^z(\Omega'))} + \|\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2\|_{L^q(\Omega' \times (0, T))} \leq C, \quad (103)$$

where  $C$  is independent of  $\varepsilon$ . Moreover, the sequences  $\varrho_\varepsilon \mathbf{u}_\varepsilon$  and  $\varrho_\varepsilon$  are equi-integrable.

*Proof.* Fix an arbitrary  $\Omega' \Subset \Omega$ . Choose a nonnegative function  $\zeta \in C_0^\infty(\mathbb{R}^2)$  with the properties:  $\zeta$  is compactly supported in  $\Omega$  and  $\zeta = 1$  in  $\Omega'$ . Notice that  $\lambda$ ,  $\zeta$ , and  $\varrho_\varepsilon$  meet all requirements of Theorem 6.1. Hence  $p(\varrho_\varepsilon)$  satisfy inequality (66). It is easy to see that estimates (102) follows from (66) and the formula  $p(\varrho) = \varrho_\varepsilon + \varrho_\varepsilon^\gamma$ . Next choose an arbitrary  $r \in (2, \infty)$  and set  $s = r/(r - \lambda) > 1$  and  $\alpha = (1 + \lambda)/r \in (0, 1)$ . Obviously

$$(1 - \alpha)/\infty + \alpha/(1 + \lambda) = 1/r, \quad 1 - \alpha + \alpha/(1 + \lambda) = 1/s.$$

From this, inequality (102), and the interpolation inequality we obtain

$$\|\varrho_\varepsilon\|_{L^r(0,T;L^s(\Omega'))} \leq \|\varrho_\varepsilon\|_{L^\infty(0,T;L^1(\Omega'))}^{1-\alpha} \|\varrho_\varepsilon\|_{L^{1+\lambda}(0,T;L^{1+\lambda}(\Omega'))}^\alpha < C, \quad (104)$$

which gives the estimate (103) for  $\varrho_\varepsilon$ . In order to estimate the quantity  $\varrho_\varepsilon|\mathbf{u}_\varepsilon|$ , represent it in the form

$$\varrho_\varepsilon|\mathbf{u}_\varepsilon| = \varrho_\varepsilon^\mu (\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2)^\beta |\mathbf{u}_\varepsilon|^\nu. \quad (105)$$

Let us show that there exist exponents  $\mu \in (1/2, 1)$ ,  $\beta, \nu \in (0, 1)$  and  $p, z, \sigma \in (1, \infty)$  with the properties

$$\begin{aligned} \beta &= 1 - \mu, \quad \nu + 2\beta = 1, \quad \text{i.e., } \nu = 2\mu - 1, \\ \mu/r + \nu/2 &= 1/p, \quad \mu/s + \beta + \nu/\sigma = 1/z. \end{aligned} \quad (106)$$

To this end notice that these relations can be equivalently rewritten in the form

$$1/p = (2\mu - 1)/2 + \mu/r, \quad 1/z = 1 + \mu(1/s - 1) + (2\mu - 1)/\sigma, \quad \beta = 1 - \mu, \quad \nu = 2\mu - 1,$$

which gives  $\mu = (1/2 + 1/p)(1 + 1/r)^{-1}$ . The inequalities  $1/2 < \mu < 1$  are fulfilled if and only if  $2r/(r + 2) < p < 2r$ . Since  $r > 2$ , there exists  $p > 2$  satisfying these inequalities. On the other hand, it follows from  $s > 1$  that  $0 < 1 + \mu(1/s - 1) < 1$ . Hence there is  $\sigma \in (1, \infty)$  such that  $z \in (1, \infty)$ . This completes the proof of the existence of exponents satisfying (106). The Hölder inequality, estimate (102), and energy estimate (19) imply

$$\begin{aligned} &\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^p(0,T;L^z(\Omega'))} \leq \\ &\|\varrho_\varepsilon^\mu\|_{L^{r/\mu}(0,T;L^{s/\mu}(\Omega'))} \|\varrho_\varepsilon^\beta |\mathbf{u}_\varepsilon|^{2\beta}\|_{L^\infty(0,T;L^{1/\beta}(\Omega'))} \| |\mathbf{u}_\varepsilon|^\nu \|_{L^{2/\nu}(0,T;L^{\sigma/\nu}(\Omega'))} \\ &= \|\varrho_\varepsilon\|_{L^r(0,T;L^s(\Omega'))}^\mu \|\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2\|_{L^\infty(0,T;L^1(\Omega'))}^\beta \|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^\sigma(\Omega'))}^\nu \leq C \|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^\sigma(\Omega'))}^\nu. \end{aligned}$$

Recall that the embedding  $W_0^{1,2}(\Omega) \hookrightarrow L^\sigma(\Omega)$  is bounded for every  $\sigma \in [1, \infty)$ . It follows from this and energy estimate (19) that

$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^\sigma(\Omega'))} \leq c \|\mathbf{u}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(\Omega))} \leq C,$$

which leads to the estimate for  $\varrho_\varepsilon \mathbf{u}_\varepsilon$  in (103). Now our task is to estimate the kinetic energy density  $\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2$ . Since  $p > 2$  and  $z > 1$  there are  $\kappa_1, \kappa_2, \omega \in$

$(1, \infty)$  such that  $1/p + 1/2 = 1/\kappa_1$  and  $1/z + 1/\omega = 1/\kappa_2$ . Set  $q = \min\{\kappa_i\}$ . Applying the Hölder inequality and using estimate (103) for  $\varrho_\varepsilon \mathbf{u}_\varepsilon$  we obtain

$$\begin{aligned} \|\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2\|_{L^q(\Omega' \times (0, T))} &\leq C \|\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2\|_{L^{\kappa_1}(0, T; L^{\kappa_2}(\Omega'))} \\ &\leq c \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^p(0, T; L^z(\Omega'))} \|\mathbf{u}_\varepsilon\|_{L^2(0, T; L^\omega(\Omega'))} \leq C \|\mathbf{u}_\varepsilon\|_{L^2(0, T; W_0^{1,2}(\Omega))} \leq C. \end{aligned}$$

This completes the proof of (103). It remains to show that the sequences  $\varrho_\varepsilon$  and  $\varrho_\varepsilon \mathbf{u}_\varepsilon$  are equi-integrable in  $Q_T$ . By energy estimate (19), the sequence  $\varrho_\varepsilon \log(1 + \varrho_\varepsilon)$  is bounded in  $L^1(Q_T)$ . Hence this sequence is equi-integrable. This means that for every  $\varkappa > 0$  there is  $\delta(\varkappa)$ , depending on  $\varepsilon$  only, such that the inequality

$$\int_A \varrho_\varepsilon dx dt \leq \varkappa$$

hold for every  $A \subset Q_T$  such that  $\text{meas } A < \delta(\varkappa)$ . By the Cauchy inequality and energy estimate (19), we have

$$\int_A \varrho_\varepsilon |\mathbf{u}_\varepsilon| dx dt \leq \left( \int_A \varrho_\varepsilon dx dt \right)^{1/2} \left( \int_{Q_T} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 dx dt \right)^{1/2} \leq E \sqrt{\varkappa}.$$

which yields the equi-integrability of the sequence  $\varrho_\varepsilon \mathbf{u}_\varepsilon$ .  $\square$

Let us turn to the proof of Theorem 1.1. It follows from energy estimates (19) and Lemma 7.1 that, after passing to a subsequence if necessary, we can assume that there are integrable functions  $\varrho$ ,  $\mathbf{u}$ ,  $\overline{\varrho \mathbf{u}}$ , and  $\overline{\varrho \mathbf{u} \otimes \mathbf{u}}$  with the properties

$$\begin{aligned} \varrho_\varepsilon \rightharpoonup \varrho, \quad \varrho_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup \overline{\varrho \mathbf{u}} \quad \text{weakly in } L^1(\Omega \times (0, T)), \\ \mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2(0, T; W_0^{1,2}(\Omega)), \end{aligned} \tag{107}$$

For every compact set  $\Omega' \subset \Omega$ , we have

$$\begin{aligned} \varrho_\varepsilon \rightharpoonup \varrho \quad \text{weakly in } L^r(0, T; L^s(\Omega')), \quad \varrho_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup \overline{\varrho \mathbf{u}} \quad \text{weakly in } L^p(0, T; L^z(\Omega')), \\ \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \rightharpoonup \overline{\varrho \mathbf{u} \otimes \mathbf{u}} \quad \text{weakly in } L^q(\Omega' \times (0, T)). \end{aligned} \tag{108}$$

Here  $r, p \in (2, \infty)$  and  $q, s, z \in (1, \infty)$  are given by Lemma 7.1. It follows from energy estimate (19) and convexity of the function  $\varrho \log(1 + \varrho)$  that  $\varrho$  and  $\mathbf{u}$  satisfies inequalities (5). Moreover,  $\varrho \in L^r(0, T; L^s(\Omega'))$ ,  $\overline{\varrho \mathbf{u}} \in$

$L^p(0, T; L^z(\Omega'))$  and  $\overline{\varrho \mathbf{u} \otimes \mathbf{u}} \in L^q(\Omega' \times (0, T))$  for ever  $\Omega' \Subset \Omega$ . Finally notice that in view of estimates (102) we have

$$\varepsilon \int_{\Omega' \times (0, T)} \varrho_\varepsilon^\gamma \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Substituting  $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$  and  $\varphi(\varrho_\varepsilon) := \varrho_\varepsilon$  into (20), (22) and letting  $\varepsilon \rightarrow 0$  we obtain that the integral identities

$$\begin{aligned} \int_{Q_T} (\overline{\varrho \mathbf{u}} \cdot \partial_t \boldsymbol{\xi} + \overline{\varrho \mathbf{u} \otimes \mathbf{u}} : \nabla \boldsymbol{\xi} + \varrho \operatorname{div} \boldsymbol{\xi} - \mathbb{S}(\mathbf{u}) : \nabla \boldsymbol{\xi}) \, dx dt \\ + \int_Q \varrho \mathbf{f} \cdot \boldsymbol{\xi} \, dx dt + \int_\Omega (\varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\xi})(x, 0) \, dx = 0 \end{aligned} \quad (109)$$

$$\int_{Q_T} (\varrho \partial_t \psi + \overline{\varrho \mathbf{u}} \cdot \nabla \psi) \, dx dt + \int_\Omega \varrho_0(x) \psi(x, 0) \, dx = 0 \quad (110)$$

hold for all vector fields  $\boldsymbol{\xi} \in C^\infty(Q_T)$  equal 0 in a neighborhood of  $\partial\Omega \times [0, T]$  and of the top  $\Omega \times \{t = T\}$  and for all  $\psi \in C^\infty(Q_T)$  vanishing in a neighborhood of the top  $\Omega \times \{t = T\}$ . It remains to prove that

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}, \quad \overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{a.e. in } Q_T \quad (111)$$

The proof is standard, see [5]. We begin with the observation that  $\varrho_\varepsilon$  and  $\mathbf{u}_\varepsilon$  satisfies the equations

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) = \operatorname{div} (\mathbb{S}(\mathbf{u}_\varepsilon) - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nabla p(\varrho_\varepsilon) + \varrho_\varepsilon \mathbf{f}, \quad \partial_t \varrho_\varepsilon = - \operatorname{div} (\varrho_\varepsilon \mathbf{u}_\varepsilon), \quad (112)$$

which are understood in the sense of the distribution theory. Notice that in view of the energy estimate (19), the sequences  $\varrho_\varepsilon$ ,  $\mathbb{S}(\mathbf{u}_\varepsilon)$ ,  $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ ,  $p(\varrho_\varepsilon)$  are bounded in the space  $L^2(0, T; L^1(\Omega))$ . Choose an arbitrary function  $\xi \in C_0^\infty(Q_T)$ . Since the embedding  $W_0^{3,2}(\Omega) \rightarrow C_0^1(\Omega)$  is bounded, it follows from (112) that the sequences  $\partial_t(\xi \varrho_\varepsilon)$  and  $\partial_t(\xi \varrho_\varepsilon \mathbf{u}_\varepsilon)$  are bounded in  $L^2(0, T; W^{-3,2}(\Omega))$ . On the other hand, Lemma 7.1 implies that the sequences  $\xi \varrho_\varepsilon$  and  $\xi \varrho_\varepsilon \mathbf{u}_\varepsilon$  are bounded in  $L^r(0, T; L^s(\Omega))$  and  $L^p(0, T; L^z(\Omega))$  respectively. Notice that  $r, p > 2$  and the embedding  $W^{-1,2}(\Omega) \hookrightarrow L^s(\Omega)$ ,  $W^{-1}(\Omega) \hookrightarrow L^z(\Omega)$  is compact for  $s, z > 1$ . Applying the Dubinskii-Lions compactness Theorem we conclude that the sequences  $\xi \varrho_\varepsilon$  and  $\xi \varrho_\varepsilon \mathbf{u}_\varepsilon$  are relatively compact in  $L^2(0, T; W^{-1,2}(\Omega))$ . From this and (107) we obtain

$$\int_{Q_T} \xi \varrho_\varepsilon \mathbf{u}_\varepsilon \, dx dt \rightarrow \int_{Q_T} \xi \varrho \mathbf{u} \, dx dt, \quad \int_{Q_T} \xi \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \, dx dt \rightarrow \int_{Q_T} \xi \varrho \mathbf{u} \otimes \mathbf{u} \, dx dt$$

as  $\varepsilon \rightarrow 0$ , which yields (111). This completes the proof of Theorem 1.1.

## 7.2 Proof of Theorem 1.2

It suffices to note that estimate (6) follows directly from Theorem 4.1, and estimates (8) follow from Proposition 5.1 and Theorem 6.1.

## A Proof of Lemmas 2.1 and 2.2

**Proof of Lemma 2.1** Introduce the polar coordinates  $\lambda = |\xi| \in \mathbb{R}^+$  and  $\omega = |\xi|^{-1}\xi \in \mathbb{S}^1$ . Applying the Fubini Theorem we obtain

$$\begin{aligned} \mathfrak{F}g(\xi) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} g(x) dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\lambda \omega \cdot x} g(x) dx = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda \tau} \left\{ \int_{\omega \cdot x = \tau} g(x) dl \right\} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda \tau} \Phi(\omega, \tau) d\tau = \frac{1}{\sqrt{2\pi}} \mathfrak{F}_\lambda \Phi(\omega, \lambda), \end{aligned}$$

where  $\mathfrak{F}_\lambda$  is the Fourier transform with respect to  $\tau$ . We thus get

$$|\mathfrak{F}g(\lambda\omega)|^2 = \frac{1}{2\pi} |\mathfrak{F}_\lambda \Phi(\omega, \lambda)|^2 \quad (113)$$

Since  $\Phi$  is a real valued function, the Plancherel identity yields

$$\int_0^\infty |\mathfrak{F}_\lambda \Phi(\omega, \lambda)|^2 d\lambda = \frac{1}{2} \int_{-\infty}^\infty |\mathfrak{F}_\lambda \Phi(\omega, \lambda)|^2 d\lambda = \frac{1}{2} \int_{-\infty}^\infty |\Phi(\omega, \tau)|^2 d\tau.$$

Integrating both sides of (113) by  $\lambda$  we conclude that

$$\int_0^\infty |\mathfrak{F}g(\lambda\omega)|^2 d\lambda = \frac{1}{4\pi} \int_{-\infty}^\infty |\Phi(\omega, \tau)|^2 d\tau$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^2} |\xi|^{-1} |\mathfrak{F}g|^2 d\xi &= \int_{\mathbb{S}^1} \int_0^\infty \frac{1}{\lambda} |\mathfrak{F}g(\lambda\omega)|^2 \lambda d\lambda d\omega = \\ &= \int_{\mathbb{S}^1} \int_0^\infty |\mathfrak{F}g(\lambda\omega)|^2 d\lambda d\omega = \frac{1}{4\pi} \int_{\mathbb{S}^1} \int_{-\infty}^\infty |\Phi(\omega, \tau)|^2 d\tau d\omega. \end{aligned}$$

Recalling expression (9) for  $H^s$ - norm we obtain the desired estimate (14).

**Proof of Lemma 2.2** Let  $s > 1/2$ . It suffices to prove that

$$\|uv\|_{H^{1/2}(\mathbb{R}^2)} \leq c\|u\|_{H^1(\mathbb{R}^2)}\|v\|_{H^s(\mathbb{R}^2)}$$

for all  $u \in H^1(\mathbb{R}^2)$  and for all  $v \in H^s(\mathbb{R}^2)$ . Choose an arbitrary  $u \in H^1(\mathbb{R}^2)$  and consider the linear operator  $\mathbf{U} : v \mapsto uv$ . Set  $\delta = s - 1/2 > 0$ . Recall that  $H^s(\mathbb{R}^2)$  coincides with  $W^{s,2}(\mathbb{R}^2)$ . Since the embedding  $H^{1+\delta}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$  is bounded, we have

$$\|uv\|_{L^2(\mathbb{R}^2)} \leq \|v\|_{L^\infty(\mathbb{R}^2)}\|u\|_{L^2(\mathbb{R}^2)} \leq c\|v\|_{H^{1+\delta}(\mathbb{R}^2)}\|u\|_{H^1(\mathbb{R}^2)} \quad (114)$$

Since the embedding  $H^{1+\delta}(\mathbb{R}^2) \hookrightarrow W^{1,2/(1-\delta)}(\mathbb{R}^2)$  and  $H^1(\mathbb{R}^2) \hookrightarrow L^{2/\delta}(\mathbb{R}^2)$  is bounded, see [1], thm. 7.57, we have

$$\begin{aligned} \|\nabla(uv)\|_{L^2(\mathbb{R}^2)} &\leq \|v\|_{L^\infty(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)} + \|u\nabla v\|_{L^2(\Omega)} \leq \\ &c\|v\|_{H^{1+\delta}}\|u\|_{H^1(\mathbb{R}^2)} + \|\nabla v\|_{L^{2/(1-\delta)}(\mathbb{R}^2)}\|u\|_{L^{2/\delta}(\mathbb{R}^2)} \leq \\ &c(\|v\|_{H^{1+\delta}(\mathbb{R}^2)} + \|v\|_{W^{1,2/(1-\delta)}(\mathbb{R}^2)})\|u\|_{H^1(\mathbb{R}^2)} \leq c\|v\|_{H^{1+\delta}(\mathbb{R}^2)}\|u\|_{H^1(\mathbb{R}^2)}. \end{aligned} \quad (115)$$

Combining (114) and (115) we obtain

$$\|\mathbf{U}v\|_{H^1(\mathbb{R}^2)} \leq c\|u\|_{H^1(\mathbb{R}^2)}\|v\|_{H^{1+\delta}(\mathbb{R}^2)}. \quad (116)$$

On the other hand, the boundedness of the embedding  $H^\delta \hookrightarrow L^{2/(1-\delta)}(\mathbb{R}^2)$  implies

$$\|uv\|_{L^2(\mathbb{R}^2)} \leq \|v\|_{L^{2/(1-\delta)}(\mathbb{R}^2)}\|u\|_{L^{2/\delta}(\mathbb{R}^2)} \leq c\|v\|_{H^\delta(\mathbb{R}^2)}\|u\|_{H^1(\mathbb{R}^2)},$$

which yields the estimate

$$\|\mathbf{U}v\|_{L^2(\mathbb{R}^2)} \leq c\|u\|_{H^1(\mathbb{R}^2)}\|v\|_{H^\delta(\mathbb{R}^2)}.$$

From this and (116) we conclude that  $\mathbf{U}$  is a bounded operator from  $H^\delta(\mathbb{R}^2)$  to  $L^2(\mathbb{R}^2)$  and from  $H^{1+\delta}(\mathbb{R}^2)$  to  $H^1(\mathbb{R}^2)$ . Moreover, its norm does not exceed  $c\|u\|_{H^1(\mathbb{R}^2)}$ . Applying the interpolation theorem, [2] Sec. 2.4, Sec. 6.4 Thm. 6.4.5, and noting that  $1/2 + \delta = s$  we obtain that the desired inequality

$$\|uv\|_{H^{1/2}(\mathbb{R}^2)} \equiv \|\mathbf{U}v\|_{H^{1/2}(\mathbb{R}^2)} \leq c\|u\|_{H^1(\mathbb{R}^2)}\|v\|_{H^{(\delta+1+\delta)/2}(\mathbb{R}^2)} = c\|u\|_{H^1(\mathbb{R}^2)}\|v\|_{H^s(\mathbb{R}^2)}$$

holds for all  $u \in H^1(\mathbb{R}^2)$  and all  $v \in H^s(\mathbb{R}^2)$ .

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